Discrete Optimization 2024 (EPFL): Problem set of week 11

May 16, 2024

1. a) What is the minimum vertex cover for the complete graph on n vertices (the graph on n vertices where every two vertices are connected by an edge)?

   b) How large can be the minimum vertex cover for a tree with n vertices?

   Solution: a) The answer is $n-1$. This is because if there are at least two vertices that we do not take for the vertex cover, there will be at least one edge that is not covered (the edge with these two vertices).

   b) A tree is a bipartite graph. The size of the minimal vertex cover is the same as that of the maximum matching. Therefore, an equivalent question is how large can be the maximum matching in a tree with n vertices. The answer is $\lfloor \frac{n}{2} \rfloor$. Clearly, it cannot be larger than this. This is attained for a path with n vertices.

2. We saw that in bipartite graph the maximum size of a matching is equal to the minimum size of a vertex cover. In general graphs the minimum vertex cover is greater than or equal to the maximum size of a matching. Show that it is always true that the minimum vertex cover is at most twice the size of the maximum matching in a graph. For every n find a graph with maximum matching equal to n and minimum vertex cover equal to $2n$.

   Solution: Assume the maximum matching has size k. Consider a matching of size k and then the $2k$ vertices of the k edges participating in that matching is a vertex cover. Indeed, if there is an edge both of whose vertices are not among the $2k$ vertices of the maximum matching, then we could add this edge to the maximum matching and get a larger matching, which is a contradiction. A graph that consists of n disjoint triangles is an example for a graph with maximum matching n and minimum vertex cover $2n$.

3. Write a linear program that finds a minimum set (if there is one) of edges of a given bipartite graph G that together contain all the vertices of G (as usual we assume G has n vertices and m edges).

   Solution: Let A be the $n \times m$ vertex-edge incidence matrix of the graph G.

   Then we look for $x_1, \ldots, x_m$ such that $x_1 + \ldots + x_m$ is minimum and $Ax \geq \overrightarrow{\mathbf{1}}$. In addition $x_i \geq 0$ (the condition $x_i \leq 1$ is now automatically satisfied). We also want x_i to be an integer but this is given for free because A is totally unimodular.

4. Write a linear program that finds a set (if there is one) of n edges in a bipartite graph G (on n vertices and m edges) that together form a union of disjoint cycles.

   Solution: A set of n edges in a graph of n vertices is a union of disjoint cycles if and only if every vertex belongs to precisely 2 of these edges.

   Let A be the $n \times m$ vertex-edge incidence matrix of the graph G. We want to find $x_1, \ldots, x_m$ such that $Ax = 2\overrightarrow{\mathbf{1}}$ and $x_i \geq 0$ and $x_i \leq 1$ (notice that here we cannot avoid the restriction $x_i \leq 1$).

   All we need is one vertex that is feasible. We do not want to maximize anything. What we can do is either ask for the maximum of $x_1 + \ldots + x_m$ under the constraint $Ax \leq 2\overrightarrow{\mathbf{1}}$. If this maximum is equal to n we are done. If not, then there is no such set.

   Another way to do it is to consider the linear program with constraints $Ax = 2\overrightarrow{\mathbf{1}}$ and $x_i \geq 0$ and $x_i \leq 1$ and look for the maximum of any linear function (for example $x_1 - 3x_4$). If the feasible set is not empty (which we find already at the beginning of the simplex algorithm once applied), then we are done.