Discrete Optimization 2024 (EPFL): Problem set of week 10

May 7, 2024

1. We are given m subsets $S_1, \ldots, S_m$ of $\{1, \ldots, n\}$.

   a) Find an integer linear program that computes the maximum number of pairwise disjoint sets among $S_1, \ldots, Sm$.

   b) Find an integer linear program that computes the minimum cardinality of a subset B of $\{1, \ldots, n\}$ such that $B \cap S_i$ is not empty for every i.

   Solution: a) Let A be the $n \times m$ matrix such that $a_{ij} = 1$ if $i \in S_j$. Otherwise $a_{ij} = 0$. Then we look for the maximum of $\langle \overrightarrow{\mathbf{1}}, x \rangle$ such that $Ax \leq \overrightarrow{\mathbf{1}}$ and $0 \leq x_i \leq 1$ and x_i is an integer.

   b) Let A be the $m \times n$ matrix such that $a_{ji} = 1$ if $i \in S_j$. Otherwise $a_{ji} = 0$. Then we look for the minimum of $\langle \overrightarrow{\mathbf{1}}, x \rangle$ such that $Ax \geq \overrightarrow{\mathbf{1}}$ and $0 \leq x_i \leq 1$ and x_i is an integer. (Notice that we may omit the condition $x_i \leq 1$ here.)

2. Let A be a matrix where column of A contains only 0's except for one coordinate that is equal to 1 and another coordinate that is equal to -1. Show that A is totally unimodular.

   Solution: We will probably see this also in class: By induction on k. For $k = 1$ this is clear. Let M be a $k \times k$ submatrix of A. If there is a row in M with no ± 1's the determinant is equal to 0. If there is a row in M with only one ± 1, we can conclude by induction on k. Assume therefore that every row of M contains at least 2 1's. Because every column of M contains at most two ± 1 it must be that every row of M contains precisely two ± 1. Now we can add the rows of M and see that their sum is equal to 0. This means the the rows of M are linearly dependent. Consequently, the determinant of A is equal to 0.

3. Give an example for a linear program with no maximum (in other words, unbounded linear program) such that the corresponding integer program is not unbounded.

   Solution: This is a bit tricky question. One can consider $\max y$ such that $\frac{1}{3} \le x \le \frac{2}{3}$ in the two dimensional plane. Clearly there is no maximum. On the other hand there are no integer feasible points at all.

   Another example is just to take a line through the origin that does not contain any integer point except for the origin (for example the line spanned by $(1, \sqrt{2})$) and maximize say the x-coordinate of a point on this line.

   It is true, however, that once there is an integer point in an unbounded polyhedron with infinite volume, then there are infinitely many integer points there (follows from problem 5).

4. Let A be a matrix where every row looks like $(0, \ldots, 0, 1, \ldots, 1, 0, \ldots, 0)$, or $(1, \ldots, 1, 0, \ldots, 0)$, or $(0, \ldots, 0, 1, \ldots, 1)$, or $(1, \ldots, 1)$. That is, all the 1's appear in one interval. Show that every $k \times k$ submatrix of A has determinant $0, 1$, or -1 (in other words, A is totally unimodular).

   Solution: By induction on the number of rows in A. If there is no 1 entry on the leftmost column, then the determinant is equal to 0. Otherwise consider all the rows with 1 on the leftmost entry. Then subtract the shortest row (the one with fewer 1's) from the other rows. We get another matrix A' with the same property and there is only one 1 entry on the first (leftmost) column. Now we conclude by induction when computing the determinant of A' that is equal to plus or minus the determinant of one principle minor of A'.

5. Let K be the cone generated by n linearly independent vectors in $\mathbb{R}^n$. Show that C must contain infinitely many integer points.

   Solution. The idea is that very far from the origin the cone has a large volume. In particular it contains a cube that is parallel to the axes and has edge length 1 (or even 1000). This cube must contain an integer point (or even 1000^n).

   To rigorously implement this idea one can take a ray that is contained in the interior of the cone (for example all positive multiples of $W = \sum_{i=1}^{n} v_i$. Then very far in the cone the cone contains a cylinder of arbitrary large radius around this ray. The cylinder will contain a large cube.