Discrete Optimization 2024 (EPFL): Problem set of week 9

April 30, 2024

Reminder: Farkas' Lemma (version I): $Ax = b$ with $x \geq 0$ has a solution iff for every $q \in \mathbb{R}^m$ such that $qA \geq 0$ we have also $\langle q, b \rangle \geq 0$.

Farkas' Lemma (version II): $Ax \leq b$ has a solution iff $q \geq 0$ and $qA = 0$, implies $\langle q, b \rangle \geq 0$.

1. Find a hyperplane separating the point $x = (1, 3, 9)$ from the cone in $\mathbb{R}^3$ generated by the three vectors $v_1 = (1, 1, 1), v_2 = (1, 2, 3)$, and $v_3 = (1, 2, 1)$.

   Solution: The cone has three facets (faces of maximal dimension). These three faces are contained in the three hyperplanes spanned by the three possible pairs of vectors from v_1, v_2, and v_3. At least one of these hyperplanes is good for us or otherwise x would belong to the cone. Now it is not difficult to find one that works.

2. Let K be a cone in $\mathbb{R}^n$. Prove that any hyper-plane H supporting K must pass through the origin O.

   Solution: Let $H = \{\langle q, x \rangle = r\}$ be a supporting hyper-plane for the cone K. We need to show $r = 0$. Let $v \in H \cap K$. We may assume that $\langle q, x \rangle \geq r$ for every $x \in K$ (otherwise replace r with $-q$ and replace r with $-r$). Because $v \in K$ and k is a cone, then also $\frac{1}{2}v, 2v \in K$. We have $\langle q, \frac{1}{2}v \rangle = \frac{1}{2}r$. and $\langle q, 2v \rangle = 2r$. Hence it must be that $\frac{1}{2}r, 2r \geq r$. This implies $r = 0$.

3. Prove that $A\overrightarrow{x} = \overrightarrow{b}$ has a solution (we do not require $x \geq 0$ as in Farkas' Lemma) if and only if for every y such that $yA = 0$ we also have $\langle y, b \rangle = 0$.

Solution: $Ax = b$ has a solution if and only if $Ax - Ay = b$ has a solution with $x, y \geq 0$. By Farkas' lemma applied for the system $A'(x, y) = b$ for $A' = (A, -A)$, this system has a solution if and only if for every q such that $qA' \geq 0$ we have $\langle q, b \rangle \geq 0$. Now $qA' \geq 0$ is equivalent to $qA \geq 0$ and $q(-A) \geq 0$. Hence $qA = 0$.

Therefore, the original system has a solution if and only if $qA = 0$ implies $\langle q, b \rangle \geq 0$.

But this is if and only if $qA = 0$ implies $\langle q, b \rangle = 0$ (because if $qA = 0$, then also $-qA = 0$ and so this should imply $\langle q, b \rangle \geq 0$ and $\langle -q, b \rangle \geq 0$. This is the same as $\langle q, b \rangle = 0$).

4. Prove the following Farkas-like Lemma: $Ax < 0, \quad x \geq 0$ has a solution if and only if there is no $y \geq 0, \quad y \neq 0$ such that $yA \geq 0$.

   Solution: $Ax < 0, x \geq 0$ has a solution if and only if $Ax + z = (-\epsilon, \ldots, -\epsilon), x \geq 0, z \geq 0$ has a solution for some $\epsilon > 0$.

   We apply Farkas' lemma. The matrix of the new system is

   $$A' = (A, I_m).$$

   The new system has a solution if and only if $qA' \geq 0$ implies also $\langle q, (-\epsilon, \ldots, -\epsilon) \rangle \geq 0$.

   Now, $qA' \geq 0$ is equivalent to $qA \geq 0$ and $q \geq 0$. In such a case in order for q to satisfy $\langle q, (-\epsilon, \ldots, -\epsilon) \rangle \geq 0$ (for some $\epsilon > 0$) we must have $q = 0$. Therefore,

   having $qA' \geq 0$ implying $\langle q, (-\epsilon, \ldots, -\epsilon) \rangle \geq 0$ is the same as $qA' \geq 0$ implying $q = 0$. This is the same as $qA \geq 0$ and $q \geq 0$ implies $q = 0$.

   This is the same as if there is no $q \geq 0, \quad q \neq 0$ such that $qA \geq 0$. May be a bit confusing to follow but this is exactly what we had to prove.

5. Prove the following Farkas-like Lemma: $Ax = 0, \quad x > 0$ has a solution if and only if there is no y such that $yA \geq 0$ and $yA \neq 0$.

   Solution: $Ax = 0, \quad x > 0$ has a solution if and only if

   $Ax = 0, -x + z = (-\epsilon, \ldots, -\epsilon), x, z \geq 0$ has a solution for some $\epsilon > 0$.

   This is the same as $A'(x, z) = (0, 0 \ldots, 0, -\epsilon, \ldots, -\epsilon)$ has a solution for the matrix

   $A' = \begin{pmatrix} A & 0_{m,n} \\ -I_n & I_n \end{pmatrix}$

By apply Farkas' lemma, this is equivalent to:

$(q, p)A' \geq 0$ implies $\langle (q, p), (0, \ldots, 0, -\epsilon, \ldots, -\epsilon) \rangle \geq 0$

This is the same as $qA - p \geq 0, \quad p \geq 0$ implies $\langle p, (-\epsilon, \ldots, -\epsilon) \rangle \geq 0$.

This is the same as $qA - p \geq 0, \quad p \geq 0$ implies $p = 0$ (because it is not possible that $p \geq 0$ and $\langle p, (-\epsilon, \ldots, -\epsilon) \rangle \geq 0$, unless $p = 0$).

This is the same as $qA \geq 0$ implies $qA = 0$ (because if not, then we could take, for example, $p = \frac{1}{2}qA$ and then $qA - p \geq 0$ but $p = \frac{1}{2}qA \neq 0$). This is exactly what we want.