Discrete Optimization 2024 (EPFL): Problem set of week 7

April 17, 2024

Reminder: The dual of the linear program $\max\{\langle c, x\rangle \mid Ax \leq b\}$ is the linear program $\min\{\langle y, b\rangle \mid yA = c, \quad y \geq 0\}$

1. Consider the linear program $\max\{\langle x, \overrightarrow{c}\rangle \mid Ax \leq b\}$ and assume that it attains a maximum at a single point x at which precisely n constraints meet. Prove that the dual linear problem has a unique minimum.

   Solution: We know from the simplex algorithm that there are $\lambda_1, \ldots, \lambda_n > 0$ such that $\sum \lambda_i a_i = c$ for some n rows of A that we assume without loss of generality are $a_1, \ldots, a_n$. Notice that $a_1, \ldots, a_n$ are linearly independent because x is a vertex.

   Consider now the vector $y = (\lambda_1, \ldots, \lambda_n, 0, \ldots, 0)$.

   Then $yA = c$ and $\langle y, b\rangle$ is the minimum of the dual problem. If there is another such point y', then y' should be positive in coordinates that are equal to 0 in y, for otherwise there is another linear combination of $a_1, \ldots, a_n$ that is equal to c. This is impossible because $a_1, \ldots, a_n$ are linearly independent.

   Now $\langle c, x\rangle = y'Ax \leq y'b$. On the other hand we assume that $y'b$ is also the minimum value of the dual program. Therefore, it must be that x satisfies equality in $Ax \leq b$ for every coordinate at which y' is positive. In particular x has to satisfy equality in another row that is not one of $a_1, \ldots, a_n$. This is a contradiction to our assumption that only n constraints meet at x.

2. What is the dual linear program to $\max\{\langle x, c\rangle \mid Ax = b\}$?

   Solution: We can write this linear program as

$$\begin{pmatrix} A \\ -A \end{pmatrix} x \leq \begin{pmatrix} b \\ -b \end{pmatrix} \tag{1}$$

The dual problem is:

$\min\{\langle u-v, b\rangle \mid (u-v)A = c, \quad u, v \geq 0\}$.

This is equivalent to $\min\{\langle y, b\rangle \mid yA = c\}$, without any condition on y being ≥ 0.

3. Let $A = I_n$ be the identity matrix.

   a) What are all the vectors c for which $\langle x, c\rangle$ has a maximum in the set $Ax \leq 0$?

   b) what is the dual linear program?

   Solution:

   a) All the vectors c with $c \geq 0$.

   b) formally it is $\min\{0 \mid y = c, y \geq 0\}$. We see that if $c \geq 0$, then there is a solution and the minimum is 0. If it is not true that $c \geq 0$, then there is no feasible point. This corresponds to having no maximum for the primal problem.

4. Let $a_1, \ldots, a_{n+1}$ be $n+1$ vectors in $\mathbb{R}^n$ such that every n of them are linearly independent.

   Show that if $\sum_{i=1}^{n+1} a_i = 0$, then for every vector c one can find nonnegative real numbers $y_1, \ldots, y_{n+1}$ such that $c = \sum_{i=1}^{n+1} y_i a_i$.

   Solution: (through duality of linear programming. There are other solutions. Could be even simpler) Let A be the matrix whose rows are $a_1, \ldots, a_{n+1}$. Notice that $Ax \leq 0$ has only the solution $x = 0$. This is because for any other x satisfying $Ax \leq 0$ it must be that Ax has some negative coordinates (why?). But this is impossible because $\overrightarrow{\mathbf{1}}A = 0$ and on the other hand $\overrightarrow{\mathbf{1}}Ax = \langle \overrightarrow{\mathbf{1}}, (Ax)\rangle$ must be negative.

   Now for every c the dual problem of $\max\{\langle x, c\rangle \mid Ax \leq 0\}$ (for which there is a maximum and it is equal to 0) is $\min\{0 \mid yA = c, y \geq 0\}$. It must be that the feasible set is not empty in order for the minimum to be 0 and not ∞. This means that we have such $y = (y_1, \ldots, y_{n+1})$ as we want.

5. What is the dual problem to the following maximization problem:

What is the maximum of $x_1 + 2x_2 + 3x_3 + \ldots + nx_n$ subject to the conditions that $x_i + x_i \leq 1$ for every $i \neq j$?

Solution: The dual problem is to find the minimum of $\sum_{1 \leq i \neq j \leq n} y_{\{i,j\}}$ where for every j we have $\sum_{i \neq j} y_{\{i,j\}} = j$ and $y_{\{i,j\}} \geq 0$ for every $i \neq j$.