Discrete Optimization 2024 (EPFL): Problem set of week 7

April 12, 2024

Reminder: The dual of the linear program $\max\{\langle c, x\rangle \mid Ax \leq b\}$ is the linear program $\min\{\langle y, b\rangle \mid yA = c, \quad y \geq 0\}$

1. Consider the linear program $\max\{\langle x, \overrightarrow{c}\rangle \mid Ax \leq b\}$ and assume that it attains a maximum at a single point x at which precisely n constraints meet. Prove that the dual linear problem has a unique minimum.

2. What is the dual linear program to $\max\{\langle x, c\rangle \mid Ax = b\}$?

3. Let $A = I_n$ be the identity matrix.

   a) What are all the vectors c for which $\langle x, c\rangle$ has a maximum in the set $Ax \leq 0$?

   b) what is the dual linear program?

4. Let $a_1, \ldots, a_{n+1}$ be $n+1$ vectors in $\mathbb{R}^n$ such that every n of them are linearly independent.

   Show that if $\sum_{i=1}^{n+1} a_i = 0$, then for every vector c one can find nonnegative real numbers $y_1, \ldots, y_{n+1}$ such that $c = \sum_{i=1}^{n+1} y_i a_i$.

5. What is the dual problem to the following maximization problem:

   What is the maximum of $x_1 + 2x_2 + 3x_3 + \ldots + nx_n$ subject to the conditions that $x_i + x_i \leq 1$ for every $i \neq j$?