Discrete Optimization 2024 (EPFL): Problem set of week 6

April 9, 2024

Reminder: The dual of the linear program $\max\{\langle c, x\rangle \mid Ax \leq b\}$ is the linear program $\min\{\langle y, b\rangle \mid yA = c, \quad y \geq 0\}$

1. Consider the following (very easy) maximizaton problem $\max\{x_1+\ldots+x_n \mid x_1, \ldots, x_n \leq 1\}$. What is the dual minimization problem?

   Solution: Our maximization problem is a linear program with $c = (1, 1, \ldots, 1)$, $A = I_n$, and $b = (1, 1, \ldots, 1)$.

   Therefore, the dual problem is to find the minimum of $y_1 + \ldots + y_n$ subject to $y_1 = y_2 = \ldots = y_n = 1$. As you can see, this is even easier than the original (very easy) maximization problem.

2. Consider the following (not very difficult) maximization problem: Find $\max \sum_{i=1}^{n} x_i$ subject to $x_i + x_j \leq 1$ for every $i \neq j$.

   What is the dual minimization problem? Try to formulate it in a natural way for a graph on n vertices since there are only n variables in dimension n.

   Solution: The linear program here is $\max\{\langle c, x\rangle \mid Ax \leq b\}$, where A is the $\binom{n}{2} \times n$ matrix with all the $m = \binom{n}{2}$ possible rows having two 1's and the rest 0's. $b = (1, 1, \ldots, 1)^T \in \mathbb{R}^{\binom{n}{2}}$ and $c = (1, 1, \ldots, 1)^T \in \mathbb{R}^n$.

   The dual problem is $\min\{\langle y, b\rangle \mid y^T A = c^T, y \geq 0\}$.

   In terms of graphs, let G be the complete graph on n vertices. It has $m = \binom{n}{2}$ edges. The dual problem is to find the optimal way to give nonnegative weights to the edges of the graph such that on one hand the sum of the weights of the edges going from any fixed vertex is equal to 1 and on the other hand we want the sum of the weights of all edges to be minimum.

Notice that for the dual problem all feasible points give the optimal value. This is not the case for the primal problem.

3. Let A be an $m \times n$ matrix with rows $a_1, \ldots, a_m$ and let $b \in \mathbb{R}^m$ be given. Consider the polyhedron P defined by $A\overrightarrow{x} \leq \overrightarrow{b}$.

   Assume that $I = \{1, 2, \ldots, n\}$ is a basis, but not a feasible basis. Denote by Q the point that is the intersection of the n hyperplanes $\{\langle a_i, x\rangle = b_i\}$ for $i = 1, \ldots, n$.

   Prove that for every $\lambda_1, \ldots, \lambda_n > 0$ there is α such that the hyperplane $H = \{\langle \sum_{i=1}^n \lambda_i a_i, x\rangle = \alpha\}$ separates Q and P.

   Solution: Because $a_1, \ldots, a_n$ are linearly independent, Q is the only point that satisfies $\{\langle a_i, x\rangle = b_i\}$ for $i = 1, \ldots, n$. Therefore, for every point of $x \in P$ we have $\langle \sum_{i=1}^n \lambda_i a_i, x\rangle < \sum_{i=1}^n \lambda_i b_i$.

   Let $y \in P$ be the point of maximum of $\langle \sum_{i=1}^n \lambda_i a_i, x\rangle$ over all x in P. We have $\langle \sum_{i=1}^n \lambda_i a_i, y\rangle < \sum_{i=1}^n \lambda_i b_i$. Taking α to be any number between $\sum_{i=1}^n b_i$ and $\langle \sum_{i=1}^n \lambda_i a_i, y\rangle$ will yield a separating hyperplane.

4. Let $\mathcal{F}$ be a family of m subsets of $\{1, \ldots, n\}$. We wish to find $x_1, \ldots, x_n$ such that $\sum x_i$ is minimum and $\sum_{i \in S} x_i \geq 1$ for every $S \in \mathcal{F}$. Verify that this problem can be written as a linear program. What is the dual (and therefore equivalent) minimization problem?

   Solution: We can write our problem as:

$$-\max\{\sum_{i=1}^{n}(-x_i) \mid Ax \leq b\},$$

   where $b = (-1, -1, \ldots, -1)^T \in \mathbb{R}^m$ and A is the matrix that has m rows that represent the sets in $\mathcal{F}$. For every $S \in \mathcal{F}$ we have a row with -1 at those coordinates $i \in S$ and 0 otherwise. Notice that $\sum_{i=1}^n(-x_i) = \langle c, x\rangle$, where $c = (-1, -1, \ldots, -1)^T \in \mathbb{R}^n$. Therefore, our linear problem is $-\max\{\langle c, x\rangle \mid Ax \leq b\}$. For convenience denote the sets in $\mathcal{F}$ by $S_1, \ldots, S_m$.

   The dual problems is $-\min\{\langle b, y\rangle \mid y^T A = c^T, \quad y \geq 0\}$. We are looking for $-\min \sum -y_i$ where $y^T A = (-1, \ldots, -1)$. This is the same as finding the maximum of $\sum y_i$ subject to $y_i \geq 0$ for every i and $\sum_{j \in S_i} y_i = 1$ for every j.

5. Consider a general linear program of the form $\max\{\langle x, c\rangle \mid Ax \leq b\}$. Assume that the vector c does not belong to the span of the rows of the matrix A (in particular A does not have a full rank n). Prove that either there is no maximum for the linear program, or there is no feasible point x satisfying $Ax \leq b$.

   Solution: There must be a vector v that is perpendicular to every row of A but not perpendicular to c. There is more than one way to see this. For example, consider the subspace of all vectors perpendicular the the span of the rows of A. Is they are all perpendicular to c it means that c is perpendicular to n linearly independent vectors (a basis of the span of the rows of A and a basis of the subspace perpendicular to the rows of A. But then c must b the 0 vector and then it belongs to the span of the rows of A.

   Take such a vector v that is perpendicular to every row of the matrix A but not perpendicular to c. Take any feasible point x (unless there is no feasible point). Then $tv + x$ is feasible for every t (why?). We can take t to be arbitrarily positive or arbitrarily negative and then $\langle c, x + tv\rangle = \langle c, x\rangle + t\langle c, v\rangle$ can be arbitrarily large or arbitrarily small. Hence there is no maximum and no minimum.