Discrete Optimization 2024 (EPFL): Problem set of week 6

April 4, 2024

Reminder: The dual of the linear program $\max\{\langle c,x\rangle \mid Ax \leq b\}$ is the linear program $\min\{\langle y,b\rangle \mid yA = c, \quad y \geq 0\}$

1. Consider the following (very easy) maximizaton problem $\max\{x_1+\ldots+x_n \mid x_1,\ldots,x_n \leq 1\}$. What is the dual minimization problem?

2. Consider the following (not very difficult) maximization problem: Find $\max \sum_{i=1}^n x_i$ subject to $x_i + x_j \leq 1$ for every $i \neq j$.

   What is the dual minimization problem? Try to formulate it in a natural way for a graph on n vertices.

3. Let A be an $m \times n$ matrix with rows $a_1,\ldots,a_m$ and let $b \in \mathbb{R}^m$ be given. Consider the polyhedron P defined by $A\overrightarrow{x} \leq \overrightarrow{b}$.

   Assume that $I = \{1,2,\ldots,n\}$ is a basis, but not a feasible basis. Denote by Q the point of intersection of the n hyper-planes $\{\langle a_i,x\rangle = b_i\}$ for $i = 1,\ldots,n$.

   Prove that for every $\lambda_1,\ldots,\lambda_n > 0$ there is α such that the hyperplane $H = \{\langle \sum_{i=1}^n \lambda_i a_i, x\rangle = \alpha$ separates Q and P.

4. Let $\mathcal{F}$ be a family of m subsets of $\{1,\ldots,n\}$. We wish to find $x_1,\ldots,x_n$ such that $\sum x_i$ is minimum and $\sum_{i\in S} x_i \geq 1$ for every $S \in \mathcal{F}$. Verify that this problem can be written as a linear program. What is the dual (and therefore equivalent) minimization problem?

5. Consider a general linear program of the form $\max\{\langle x,c\rangle \mid Ax \leq b\}$. Assume that the vector c does not belong to the span of the rows of the matrix A (in particular A does not have a full rank n). Prove that either there is no maximum for the linear program, or there is no feasible point x satisfying $Ax \leq b$.