Discrete optimization 2024 (EPFL): Problem set of week 5

March 26, 2024

1. Consider the simplex (tetrahedron) P in $\mathbb{R}^3$ whose vertices are $A = (1,0,0)$, $B = (-1,1,0)$, $C = (-1,-1,0)$, and $D = (0,0,1)$. Find all the vectors $\overrightarrow{c}$ such that the maximum of $\langle \overrightarrow{c}, \overrightarrow{x} \rangle$ on P is at the vertex $(0,0,1)$.

   Solution. One possible solution is direct: write $\overrightarrow{c} = (x,y,z)$. We know that the maximum is always obtained at a vertex. Therefore, we need that $z \geq x$, $z \geq y - x$, and $z \geq -x - y$. From the last two inequalities we get $2z \geq -2x$. Now together with the first inequality this implies $z \geq 0$. If $z = 0$, then necessarily $x = 0$ and $y = 0$. We get the vector $(0,0,0)$. This is a "trivial" solution. If $z > 0$ we may assume that it is equal to 1. This is because for any solution $\overrightarrow{c}$ also a positive multiple of it will work. We now get $y - x \leq 1$ and $-x - y \leq 1$. This gives $x \geq -1$. For every $x \geq -1$ we need $-(1+x) \leq y \leq 1+x$ Therefore, the solution are all the positive multiples of $(x,y,1)$ where $x \geq -1$ and $-(1+x) \leq y \leq 1+x$.

   A more generic solution is to find the three vectors that are orthogonal to the three hyper-planes meeting at $(0,0,1)$ and pointing outside of P. Denoting these vectors by v_1, v_2, and v_3, there are b_1, b_2, b_3 such that every point x in P satisfies $\langle \overrightarrow{x}, v_i \rangle \leq b_i$ for $i = 1.2.3$. Denote by H_i the hyper-plane $\langle \overrightarrow{x}, v_i \rangle = b_i$.

   Any vector c can be written in one way as $\overrightarrow{c} = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3$. If $\alpha_1, \alpha_2, \alpha_3$ are nonnegative, then $\overrightarrow{c}$ will work for us because $(0,0,1)$ has the maximal scalar product with each of v_1, v_2, and v_3. If one of $\alpha_1, \alpha_2, \alpha_3$ is negative, say $\alpha_1 < 0$, then $\overrightarrow{c}$ is not a solution because by moving from $(0,0,1)$ in a direction that is "away from" (makes an obtuse angle with) v_1 along the intersection of H_2 and H_3, will increase the scalar product with $\overrightarrow{c}$.

The solution is therefore, any positive linear combination of v_1, v_2, and v_3.

2. Let P be the tetrahedron whose vertices are $A = (1, 2, 3)$, $B = (2, 1, -1)$, $C = (1, 1, 0)$, and $D = (2, 1, -3)$. Find all the vectors $\overrightarrow{c}$ such that the function $\langle \overrightarrow{c}, x \rangle$ is maximized at every point on the edge AC and at no other point.

   Solution. We first find a vector v_1 orthogonal to the hyperplane H_1, through A, C and B, and a vector v_2 orthogonal to the hyperplane H_2, through A, C and D. Then the answer is any linear combination of v_1 and v_2 with strictly positive coefficients.

3. Let $P \subset \mathbb{R}^n$ be the polytope define by the inequalities $x_i \geq 0$ for $i = 1, \ldots, n$ and $a_1x_1 + a_2x_2 + \ldots + a_nx_n \leq 1$ for every vector $(a_1, \ldots, a_n)$ whose coordinates are a permutation of the numbers $1, \ldots, n$. Find all the neighbors of the vertex O of P.

   Solution. It is not hard to check that the lines containing the edges coming out from P are the lines of the axes spanned by the unit vectors $e_1, \ldots, e_n$, respectively. Because of the symmetry of the problem, let us consider the edge contained in the line spanned by e_1.

   We now check what are the intersection points of the hyperplanes defining the other facets of P with this line. Consider the hyperplane $\{x \in \mathbb{R}^n : a_1x_1 + \ldots + a_nx_n = 1\}$. It intersects with the line spanned by e_1 at the point $(\frac{1}{a_1}, 0, \ldots, 0)$. Therefore, the closest intersection point to O is the point $(\frac{1}{n}, 0, \ldots, 0)$. This is one neighbor of O. The other neighbors are all the other vectors whose coordinates are all equal to 0 except for one that is equal to $\frac{1}{n}$.

4. Let P be the unit cube in $\mathbb{R}^n$. That is $P = \{(x_1, \ldots, x_n) \mid 0 \leq x_i \leq 1, \quad i = 1, \ldots, n\}$. Show that for every $\overrightarrow{c} \in \mathbb{R}^n$ the simplex algorithm will find the maximum of $\langle \overrightarrow{c}, \overrightarrow{x} \rangle$ over all $\overrightarrow{x} \in P$ in at most n iterations (although it has 2^n vertices).

   Solution: Notice that the vertices of the cube are precisely the 2^n vectors $\{(x_1, \ldots, x_n) \mid x_i \in \{0, 1\}\}$. Two vertices are neighbors (adjacent) if and only if they differ in only one coordinate. This is because every $n - 1$ supporting hyperplanes with nonempty intersection fix $n - 1$ of the coordinates of the two vertices of the same edge of the cube. When we apply an iteration of the simplex algorithm, we move from one vertex to a neighbor of it. We say that the index of this iteration is i if the two vertices differ at the i'th coordinate. We claim that we never have two iterations with the same index. This is because if we have an iteration with index i then c_i must be different from 0 or otherwise this iteration will not improve the value of $\langle \overrightarrow{c}, \overrightarrow{x} \rangle$. If $c_i > 0$, for example, then in this iteration we necessarily change x_i from 0 to 1. This can happen only once. We will never change x_i from 1 to 0 again because it will not improve $\langle \overrightarrow{c}, \overrightarrow{x} \rangle$. Therefore, it cannot be that we will have another iteration with index i. We argue similarly if $c_i < 0$. It now follows that there are at most n iterations.

5. For a polytope P it is known that $(0, 0, 0)$ is a vertex of P and its only neighbors are $A = (1, 2, 3)$, $B = (1, 1, 1)$, and $C = (3, 0, 1)$. Find all the vectors $\overrightarrow{c}$ such that the only improving step when maximizing $\langle \overrightarrow{c}, x \rangle$ with the simplex algorithm if we start at $(0, 0, 0)$ is to move to $(1, 1, 1)$.

   Solution. We first find three normal vectors to the three facets meeting at $(0, 0, 0)$ pointing outside of P. We denote these vectors by v_A, v_B, and v_C. v_A is orthogonal to the hyperplane through O, B, and C and similarly we take v_B and v_C.

   The answer is then that we can take c to be any linear combination of v_A, v_B, and v_C such that the coefficients of v_A and v_C are either non-negative non-negative and the coefficient of v_B is negative or the non-negative coefficient of v_A or v_C is dominating the evaluation of $c^T x$ for all $x \in P$ such that $(1, 1, 1)$ is the only vertex that is maximized.