# Discrete Optimization 2024 (EPFL): Problem set of week 5

March 26, 2024

1. Consider the simplex (tetrahedron) $P$ in $\mathbb{R}^3$ whose vertices are $A = (1, 0, 0)$, $B = (-1, 1, 0)$, $C = (-1, -1, 0)$, and $D = (0, 0, 1)$. Find all the vectors $\overrightarrow{c}$ such that the maximum of $\langle \overrightarrow{c}, \overrightarrow{x} \rangle$ on $P$ is at the vertex $(0, 0, 1)$.

2. Let $P$ be the tetrahedron whose vertices are $A = (1, 2, 3)$, $B = (2, 1, -1)$, $C = (1, 1, 0)$, and $D = (2, 1, -3)$. Find all the vectors $\overrightarrow{c}$ such that the function $\langle \overrightarrow{c}, x \rangle$ is maximized at every point on the edge $AC$ and at no other point.

3. Let $P \subset \mathbb{R}^n$ be the polytope define by the inequalities $x_i \geq 0$ for $i = 1, \ldots, n$ and $a_1x_1 + a_2x_2 + \ldots + a_nx_n \leq 1$ for every vector $(a_1, \ldots, a_n)$ whose coordinates are a permutation of the numbers $1, \ldots, n$. Find all the neighbors of the vertex $O$ of $P$.

4. Let $P$ be the unit cube in $\mathbb{R}^n$. That is $P = \{(x_1, \ldots, x_n) \mid 0 \leq x_i \leq 1 \;\; i = 1, \ldots, n\}$. Show that for every $\overrightarrow{c} \in \mathbb{R}^n$ the simplex algorithm will find the maximum of $\langle \overrightarrow{c}, \overrightarrow{x} \rangle$ over all $\overrightarrow{x} \in P$ in at most $n$ iterations (although it has $2^n$ vertices).

5. For a polytope $P$ it is known that $(0, 0, 0)$ is a vertex of $P$ and its only neighbors are $A = (1, 2, 3)$, $B = (1, 1, 1)$, and $C = (3, 0, 1)$. Find all the vectors $\overrightarrow{c}$ such that the only improving step when maximizing $\langle \overrightarrow{c}, x \rangle$ with the simplex algorithm if we start at $(0, 0, 0)$ is to move to $(1, 1, 1)$.