Discrete Optimization 2024 (EPFL): Problem set of week 4

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1. Let A be the matrix
$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \\ -1 & -1 & -1 \end{pmatrix}.$$
Let $\overrightarrow{b} = (1,1,1,1,1)$ and let $P = \{\overrightarrow{v} = (x,y,z) \in \mathbb{R}^3 \mid A\overrightarrow{v} \leq \overrightarrow{b}\}$. Show that P is a bounded polytope and find all its vertices.

   What is the maximum value of $x + 2y + 3z$ on P?

   Solution. P is clearly a polyhedron. To see that it is bounded (and therefore, a polytope) observe that by the first three rows of A we get for every point $(x,y,z) \in P$ that $x,y,z \leq 1$. Moreover, $-x-y-z \leq 1$ and consequently $x \geq -z-y-1 \geq -3$. Similarly, $y,z \geq -3$. This shows that P is bounded.

   To find the vertices of P we notice that three linearly independent rows of A are every three that do not contain the last two rows together. This leaves 7 options. These are the first three rows, the forth row and any two of the first three (there is a symmetry here) and finally, the last row and any two of the first three (here too there is symmetry).

   The first three rows of A suggest the vertex $(1,1,1)$. However, this point is not feasible, as it does not satisfy the inequality of the forth line of A.

   Considering the forth row of A and the first two, we get the point $(1,1,-1)$. This is a feasible point and therefore a vertex. Similarly, $(1,-1,1)$ and $(-1,1,1)$ are vertices.

Considering the fifth row of A and the first two, we get the point $(1,1,-3)$. This is a feasible point and therefore a vertex. Similarly, $(1,-3,1)$ and $(-3,1,1)$ are vertices.

These are all the six vertices of P.

2. Let A be the $2^n \times n$ matrix whose rows are all the 2^n possible combinations of 1 and -1. Let $\overrightarrow{b} = (1,1,1,\ldots,1) \in \mathbb{R}^{2^n}$.

   Show that $\{\overrightarrow{x} \mid A\overrightarrow{x} \leq \overrightarrow{b}\}$ is a polytope and find all its vertices.

   Solution. Observe that if $(x_1,\ldots,x_n) \in P$, then necessarily $|x_1| + \ldots + |x_n| \leq 1$. This is because we can choose the inequality defined by the row of A in which the $+1$'s and -1's correspond to the signs of the coordinates $x_1,\ldots,x_n$. In particular, P is bounded and therefore it is a polytope.

   We notice also that if $|x_1| + \ldots + |x_n| \leq 1$. then $(x_1,\ldots,x_n)$ is in P. Therefore, P is precisely the set of all points $(x_1,\ldots,x_n)$ with $|x_1| + \ldots + |x_n| \leq 1$. Therefore, any point on the boundary of P and in particular the vertices of P must satisfy $|x_1| + \ldots + |x_n| = 1$.

   If $(x_1,\ldots,x_n)$ is a point with $|x_1| + \ldots + |x_n| = 1$ and **both** $|x_i|$ **and** $|x_j|$ **are greater than** 0, then we may increase the absolute value of x_i and at the same time decrease by the same amount the absolute value of x_j and remain in P. We can also do the opposite operation and remain in P. Hence, there is a vector v such that both $\overrightarrow{x} + \overrightarrow{v}$ and $\overrightarrow{x} - \overrightarrow{v}$ are in P. This implies that $\overrightarrow{x}$ cannot be a vertex. There cannot be a supporting hyperplane for P that meet P only at $\overrightarrow{x}$.

   We conclude that the only possible candidates for vertices are the $2n$ points $(x_1,\ldots,x_n)$ with all the coordinates being equal to 0 except one coordinate that is equal either to 1 or -1. Because of symmetry, they are all vertices.

3. Let A be the matrix $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 2 & 3 \end{pmatrix}$. Let $b = (0,0,0,-6)$ and let $P = \{x \in \mathbb{R}^3 \mid Ax \leq b\}$. Find all the vertices of P and for each vertex find a supporting hyperplane.

   Solution: The vertices are $(-6,0,0), (0,-3,0)$, and $(0,0,-2)$. For each vertex consider the three facets meeting at this vertex and as we did in class take as a normal vector to the supporting hyperplane the sum of the three normals to the three facets. For example, a supporting hyperplane at the vertex $(-6,0,0)$ is $\{x+3y+4z=-6\}$

4. Let $P \subset \mathbb{R}^n$ be the cube defined by
   $P = \{(x_1,\ldots,x_n) \mid -1 \le x_1,\ldots,x_n \le 1\}$.

   a) Find a matrix A and a vector b such that $P = \{\overrightarrow{x} \mid A\overrightarrow{x} \le \overrightarrow{b}\}$.

   b) Show that the vertices of P are precisely all the 2^n points $(\pm1,\pm1,\ldots,\pm1)$.

   Solution: a) Take $A = \begin{pmatrix} I_n \\ -I_n \end{pmatrix}$. Then take $b = (1,1,\ldots,1) \in \mathbb{R}^{2n}$.

   b) Taking any n linearly independent rows of the matrix A and solving $A_I x = b_I$ clearly gives only solutions of the form $(\pm1,\pm1,\ldots,\pm1)$. To see that each such point is indeed a vertex let $\overrightarrow{v}$ be a vector whose coordinates are all equal to either $+1$ or -1. Then take $c = \overrightarrow{v}$ and observe that is $x \in P$, then $\langle c, x\rangle \le n$ because each coordinate of c is either $+1$ or -1 and each coordinate of x is between -1 and 1. Moreover, equality is possible only if every coordinate of x is equal to the corresponding coordinate of c, that is if $x = \overrightarrow{v}$. Indeed we notice that $\langle c, \overrightarrow{v}\rangle = \langle \overrightarrow{v}, \overrightarrow{v}\rangle = n$.