Discrete Optimization 2024 (EPFL): Problem set of week 3

March 9, 2024

1. A set K in $\mathbb{R}^n$ is called *convex* if for every x and y in K and for every $0 \leq \lambda \leq 1$ also the point $\lambda x + (1-\lambda)y$ is in K. In other words, the entire line segment with endpoints x and y is in K.

   a) Prove that any intersection of convex sets is convex.

   b) For any given $\overrightarrow{a} \in \mathbb{R}^n$ and any $b \in \mathbb{R}$ prove algebraically (from the algebraic definitions) that the half-space $\{\overrightarrow{x} \mid < \overrightarrow{x}, \overrightarrow{a} > \leq b\}$ is convex.

   c) Conclude from a) and b) that every intersection of half-spaces is convex.

2. Let Q be the quadrangle in the plane whose vertices are $(4,3)$, $(3,4)$, $(2,3)$, and $(3,2)$. Find a matrix A and a vector $\overrightarrow{b}$ such that $Q = \{\overrightarrow{v} \mid A\overrightarrow{v} \leq \overrightarrow{b}\}$.

3. Let B be the box in $\mathbb{R}^3$ defined by $B = \{\overrightarrow{v} = (x,y,z) \mid 1 \leq x \leq 5, \quad -2 \leq y \leq 6, \quad 0 \leq z \leq 2\}$. Find a matrix A and a vector $\overrightarrow{b}$ such that $B = \{\overrightarrow{v} = (x,y,z) \mid A\overrightarrow{v} \leq \overrightarrow{b}\}$.

4. Let P be the three dimensional pyramid with vertices $(1,1,-6)$, $(1,3,-4)$, $(-1,-2,5)$, and $(3,5,1)$. Find $\overrightarrow{c} \in \mathbb{R}^3$ such that the function $\langle \overrightarrow{c}, (x,y,z) \rangle$ attains its maximum on P precisely at the vertex $(1,3,-4)$.

5. Let P be the polyhedron defined by $P = \{v \mid A\overrightarrow{x} \leq \overrightarrow{b}\}$. Assume that $v_1, \ldots, v_k$ are k points in P and $v_1 + \ldots + v_k = 0$. Show that $0 \in P$.