Discrete Optimization 2024 (EPFL): Problem set of week 9

April 28, 2024

Reminder: Farkas' Lemma (version I): $Ax = b$ with $x \geq 0$ has a solution iff for every $q \in \mathbb{R}^m$ such that $qA \geq 0$ we have also $\langle q, b \rangle \geq 0$.

Farkas' Lemma (version II): $Ax \leq b$ has a solution iff $q \geq 0$ and $qA = 0$, implies $\langle q, b \rangle \geq 0$.

1. Find a hyperplane separating the point $x = (1, 3, 9)$ from the cone in $\mathbb{R}^3$ generated by the three vectors $v_1 = (1, 1, 1), v_2 = (1, 2, 3)$, and $v_3 = (1, 2, 1)$.

2. Let K be a cone in $\mathbb{R}^n$. Prove that any hyper-plane H supporting K must pass through the origin O.

3. Prove that $A\overrightarrow{x} = \overrightarrow{b}$ has a solution (we do not require $x \geq 0$ as in Farkas' Lemma) if and only if for every y such that $yA = 0$ we also have $\langle y, b \rangle = 0$.

4. Prove the following Farkas-like Lemma: $Ax < 0, \;\; x \geq 0$ has a solution if and only if there is no $y \geq 0, \;\; y \neq 0$ such that $yA \geq 0$.

5. Prove the following Farkas-like Lemma: $Ax = 0, \;\; x > 0$ has a solution if and only if there is no y such that $yA \geq 0$ and $yA \neq 0$.