Discrete Optimization 2024 (EPFL): Problem set of week 4

March 14, 2024

1. Let A be the matrix
   $$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \\ -1 & -1 & -1 \end{pmatrix}.$$
   Let $\overrightarrow{b} = (1,1,1,1,1)$ and let $P = \{\overrightarrow{v} = (x,y,z) \in \mathbb{R}^3 \mid A\overrightarrow{v} \leq \overrightarrow{b}\}$. Show that P is a bounded polytope and find all its vertices.

   What is the maximum value of $x + 2y + 3z$ on P?

2. Let A be the $2^n \times n$ matrix whose rows are all the 2^n possible combinations of 1 and -1. Let $\overrightarrow{b} = (1,1,1,\ldots,1) \in \mathbb{R}^{2^n}$.

   Show that $\{\overrightarrow{x} \mid A\overrightarrow{x} \leq \overrightarrow{b}\}$ is a polytope and find all its vertices.

3. Let A be the matrix $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 2 & 3 \end{pmatrix}$. Let $b = (0,0,0,-6)$ and Let $P = \{x \in \mathbb{R}^3 \mid Ax \leq b\}$. Find all the vertices of P and for each vertex find a supporting hyperplane.

4. Let $P \subset \mathbb{R}^n$ be the cube defined by
   $P = \{(x_1,\ldots,x_n) \mid -1 \leq x_1,\ldots,x_n \leq 1\}$.

   a) Find a matrix A and a vector b such that $P = \{\overrightarrow{x} \mid A\overrightarrow{x} \leq \overrightarrow{b}\}$.

   b) Show that the vertices of P are precisely all the 2^n points $(\pm 1, \pm 1, \ldots, \pm 1)$.