Graph Theory 2023 (EPFL): Problem set of week 14

December 21, 2023

1. Let G be a graph on n vertices. Show that the edges of G can be partitioned into at most $n/2$ tours. The tours can be closed or not but no two tours can share an edge.

   Solution: We may assume that G is connected, or else we work on each connected component separately. Observe that G must have an even number of vertices of odd degree. This is because the sum of all degrees is an even number. Add an artificial x vertex to G and connect it to all vertices that have odd degree in G. We get a new graph G' that is connected and all its degrees are even. Use Euler's theorem to find Euler cycle in G'. Then removing x we split the cycle into at most $n/2$ paths (in fact it will be half the number of vertices with odd degree in G) that include all edges in G.

2. Let M be a planar map representing a crossing free drawing of a planar bipartite graph. Prove that there is a closed curve in the plane that crosses every edge of M precisely once.

   Solution: Because M is bipartite, every face in M must have an even number of edges. We define a graph on the faces of M. We choose an arbitrary point inside a face as a vertex to represent this face and we draw an edge between two vertices if the corresponding faces that have a common edge in M (more precisely, we draw an edge between two faces for every common edge they have and so two faces may be connected by more than one edge). Because every face in M has an even number of edges, the resulting new graph has all of its degrees even. It is easily seen to be connected (we can reach every face from every other face by moving between two neighboring faces). An Euler cycle in the new graph is precisely the curves that we are looking for.

3. Let G be a graph. The edge-graph of G that we denote by G' is a graph whose vertices are the edges of G. Two vertices of G' are connected by an edge if the corresponding edges in G share a vertex.

   Show that if G has Euler cycle, then G' has a Hamilton cycle and also an Euler cycle.

   Solution: To see that G' has a Hamilton cycle, just follow the Euler cycle in G. It contains all the edges in G and move from an edge to another edge that has a common vertex with it. This is precisely a Hamilton cycle in G'. Now observe that the degree of each vertex in G is even. Therefore, every edge in G is connected to an even number of other edges in G. It follows now that all the degrees in G' are even. G' is also connected and so it has an Euler cycle.

4. Let G be a graph on n vertices with $(n-1)(n-2)/2+2$ edges. Show that G has a Hamilton cycle. Give an example for a graph with n vertices and $(n-1)(n-2)/2+1$ edges that does not have a Hamilton cycle.

   Solution: We claim that for every two nonadjacent vertices x and y in G we have $d(x)+d(y) \geq n$ (and then G has a Hamilton cycle by a theorem we proved in class). Indeed, otherwise there are two non-adjacent vertices x and y that together have at most $n-1$ edges incident to them. Then the maximum possible number of edges in G would be $\binom{n}{2}-(2n-3)+(n-1)=(n-1)(n-2)/2+1$, a contradiction.

   The complete graph on $n-1$ vertices together with an extra vertex connected by a single edge to another vertex is an example of a graph with $(n-1)(n-2)/2+1$ vertices and no Hamilton cycle (why?).

5. n tennis players play $\binom{n}{2}$ games with one another so that every two play once. Prove that it is always possible to arrange the people in a row such that every person (except for the leftmost) won the person standing to its left.

   Solution: Consider the longest possible such row of people $x_1x_2\dots x_k$. We will show that it contains all people (that is $k=n$). Assume not and consider a person y that is not there. It must be that x_k won y, or else we could place y to the right of x_k and get a longer row. Similarly, it must be that y won x_1, or else we could place y to the left of x_1 and get a longer row. It follows now (by kind of continuity argument) that there must be i such that x_i won y and y won x_{i-1} (why?). We can now take the longer row $x_1 x_2 \ldots x_{i-1} y x_i \ldots x_k$. This is a contradiction to this we started with the longest row possible.