Graph Theory 2023 (EPFL): Problem set of week 13

December 14, 2023

1. Show that for any k there is $n(k)$ such that in any set of $n > n(k)$ points in $\mathbb{R}^3$ either there are k points on the same 2-dimensional plane, or there are k points no 4 of them lie on a common plane.

   Proof: We make a 4-uniform hypergraph with vertices denoting the k points. We add a hyperedge to each 4-tuple of the k points. Then we take $n(k) = R_4(k,k)$. Given $n > n(k)$ points in $\mathbb{R}^3$, we color all the 4-tuples of points (hyperedges!) in two colors red and blue. We color a 4-tuple of points (a hyperedge!) blue if the 4 vertices inside the tuple are contained in a plane. Otherwise, we color the hyperedge red.

   Then we can either find k points such that all the hyperedges are red (which is precisely k points such that no 4 of them lie on a common plane). If we find k points such that every 4 of them lie on a common plane (so all hyperedges going between the k points are blue), then we claim that in this case all the points lie on a common plane. Indeed, assume there are three points A, B, C among them that are not collinear. Then the unique plane passing through A, B, C must also contain every other point from the set. If every three points are collinear, then all the points lie on one line and in particular on one (not unique) plane.

2. Show that for every k there is $n(k)$ such that if $n > n(k)$ and we color the set of all rational numbers $\frac{a}{b}$ such that $1 \leq a < b \leq n$ by k colors, then one can find a monochromatic triple of such rational numbers x, y, z such that $xy = z$.

   Proof. Similar to Schur's theorem. We take $n(k) = R_k(3, 3, \ldots, 3)$. We color the edges of the complete graph on the vertices $1, \ldots, n$ by k colors in the following way. For $i < j$ we color (i, j) by the color of the rational number $\frac{i}{j}$.

Then we can find three numbers $i < j < \ell$ such that all pairs are say red. This means in particular that $x = a/b, y = b/c$, and $z = a/c$ are all rational numbers colored red in the original coloring. We have $xy = z$.

3. Let G be an infinite graph. That is, a graph on a set of vertices that is infinite. Prove that if G is connected (there is a path between any two vertices), then either there is a vertex of infinite degree in G, or there is an infinite path in G (could be that both exist).

   Proof. If there is a vertex of infinite degree, we are done. Otherwise, assume every vertex has only finite degree. Start from a vertex v_1. It has finitely many neighbors $a_1, \ldots, a_k$. For each i let A_i denote the set of all vertices that are reachable by a simple (without recurrent vertices) path starting from a_i that avoids v_1. One of the sets A_i must be infinite, because G is infinite and connected. We rename a_i as v_2 and we consider only the subgraph of G that consist of the union of all simple paths starting as v_2 and not containing v_1. We now continue this way with v_2 in the role of v_1. We find $v_3, \ldots$ and thus generate an infinite path.

4. Let k be fixed. Prove that for any coloring of the two dimensional integer grid points (these are points of the form (a, b), where both a and b integers) with k one can find integers $x_1 < \ldots < x_{100}$ and $y_1 < \ldots < y_{100}$ such that all the points (x_i, y_j) have the same color.

   Proof. There is more than one way to do this. We will present a proof that is more related to Ramsey. Consider the diagonal grid points (i, i) where i is an integer. We consider the (infinite graph) on the set of integers and for $i < j$ we color the edge between i, j by the color of the point (i, j). We get a coloring of the complete infinite graph with k colors. We can find there a large monochromatic (say red) set. Larger than 200 (in fact we can find infinite monochromatic set).

   These 200 vertices we write as $x_1 < x_2 < \ldots < x_{100} < y_1 < y_2 < \ldots < y_{100}$. Then for every $1 \leq i, j \leq 100$ we have (x_i, y_j) is colored red, as desired.