Graph Theory 2023 (EPFL): Problem set of week 12

December 7, 2023

1. Prove that in any coloring of the edges of K_6 with two colors there are at least **two** monochromatic triangles.

   Solution: The number of possible triangles in K_6 is $\binom{6}{3} = 20$. Every triangle that is not monochromatic contains precisely two bi-chromatic angles (two edges with different colors meeting at a vertex. How many times can a specific vertex serve as a vertex of a bi-chromatic angle? At most 6 times. This is if two of the edges incident to this vertex have one color and the other three a different color. Therefore, the number of bi-chromatic angles is at most 36. This means that there are at most 18 bi-chromatic triangles. Therefore, there are at least two monochromatic triangles.

2. Show that for every r there exists $n(r)$ such that for any set of $n > n(r)$ points in the plane there exists a coloring of the $\binom{n}{2}$ segments connecting the n points by two colors there is a monochromatic path of length r going from left to right (in the sense that it is advancing) and all of whose edges are with positive slopes, or all of whose edges are with negative slopes.

   Solution: Take $n(r) = R(r, r)$. Given any set of $n > n(r)$ points in the plane, color with blue all the segments with positive slope and color with red all the segments with negative slope. By Ramsey's theorem and because $n > R(r, r)$ there are r points such that all the segments between two of them are monochromatic. If we order the points from left to right and consider the path composed of edges between consecutive points we get the desired path.

3. We are given a set of n segments in the plane. It is known that no 100 segments may be pairwise intersecting. Prove that if n is large enough,

then one can find among the segments 200 segments that are pairwise disjoint.

Solution: Take $n > R(100, 200)$ and define a graph on the n segments. Color an edge between two segments blue if they intersect and color the edge red if they are disjoint. Then either there are 100 segments such that every pair intersect, or 200 segments no two of which intersect. Because we know the first option is impossible (this is given to us), it must be that there are 200 segments no two of which cross.

4. Show that for every r there is $n(r)$ such that for every $n > n(r)$ and any coloring of the edges of the complete bipartite graph $K_{n,n}$ in two colors, there is a monochromatic complete bipartite subgraph $K_{r,r}$.

   Solution: We have seen in class the Kovari-Sos-Turan theorem: a graph with n vertices and more than $c_r n^{2-\frac{1}{r}}$ edges must contain a copy of $K_{r,r}$, where c_r is a constant depending only on r.

   Let n be so large that $\frac{1}{2}n^2 \geq c_r(2n)^{2-\frac{1}{r}}$. For this to be true we need: $n > 4(2c_r)^r$. Then in any coloring of the edges of $K_{n,n}$ either the number of red edges, or the number of blue edges is greater than $\frac{1}{2}n^2$. This in turn is greater than $c_r(2n)^{2-\frac{1}{r}}$. By the Kovari-Sos-Turan theorem it contains a complete bipartite subgraph $K_{r,r}$ that is of course monochromatic.

5. Prove that if n is large enough, then no matter how we place n points in the plane, one can always find 1000 pairwise crossing segments connecting pairs of our points.

   Solution: We have seen in class (hopefully and if not we see it next week) that if n is large enough we can find 2000 points in convex position. Then if we connect "opposite points" among them by segments we get 1000 pairwise crossing segments.