# Graph Theory 2023 (EPFL): Problem set of week 11

December 1, 2023

1. At EPFL 100 students stay over Christmas on campus. Santa Claus brings over 10,000 presents for the 100 students. The presents are of 100 types where of each type there are 100 units. Santa Claus wanted each student to take one present of each type but the students at EPFL drank too much wine that night and each of them took randomly 100 of the presents. Next morning they realized their mistake and wanted to fix it. Santa Claus allowed them to switch presents among themselves (any two students can switch presents with one another, you give $k$ presents and get $k$ presents) but only in such a way that if you get a present from someone it remains yours forever and you cannot switch it anymore.

   Show that if the students are very clever (and they are because they are EPFL students) they can switch the presents in such a way that at the end each student has all the 100 types of presents.

   Hint: It is not such an easy puzzle but it has a one line solution if you recall one of the puzzles we did in class today.

   Solution: Think of a $100 \times 100$ matrix with the numbers 1 to 100 written in it such that every number appears precisely 100 times. We can think about each column as the set of presents that a student has at the beginning.

   We have seen in class that we can permute each column in such a way that every row contains all the numbers $1, 2, \ldots, 100$ in some order. Of course permuting each column does not change anything for our problem. But now flip (take the transpose of) the matrix so that the rows become the columns. Now every student has all the 100 types of presents. Flipping the matrix corresponds to pairs of students exchanging presents. In fact every two students exchange precisely one present with each other. Nice!

2. In a $10 \times 10$ matrix all the numbers from 1 to 100 are written in an arbitrary way. At every step we are allowed to permute all the rows, or all the columns. Our goal is to reach the situation where the numbers are in order from left to right and from top to bottom: the first row is $1, 2 \ldots, 10$. The second row is $11, 12, \ldots, 20$, and so on. What is the minimum number $k$ of steps that can guarantee that we can always reach our goal in at most $k$ steps?

   Solution: The answer is $k = 3$. To show that $k = 3$ steps are enough replace each of $1, 2 \ldots, 10$ by 1. Similarly, replace each of $11, \ldots, 20$ by 2 and so on.

   We know that we can permute each row such that each of the columns contain all the numbers $1, \ldots, 10$ in some order. Now permute each column such that the first row contains only the number 1, the second row only the number 2 and so on. Going back to the original values of the entries we see that if we just permute each row now we get the desired ordering of the numbers in the matrix.

   To see that one cannot do it in less than 3 steps just consider the $2 \times 2$ matrix that has 1 and 3 in the first row and 2 and 4 in the second row.

3. Let $G$ be a bipartite graph that is $k$-regular. That is, the degree of every vertex in $G$ is equal to $k$. Prove that the edges in $G$ can be partitioned into $k$ perfect matchings.

   Solution: Notice that it must be that the two "sides" of the graph have equal size. Otherwise the graph cannot be $k$-regular and bipartite. It is not difficult to show that Hall's condition is satisfied. It also follows from a problem on the previous problem set. After we get one perfect matching and remove it from the graph we get a $(k-1)$-regular graph and continue in the same manner.

4. Show that if $G$ any graph (not necessarily bipartite, then the size of the minimum vertex cover is at most two times the size of the maximum matching. Find examples with arbitrary number of vertices where it is equal to two times the size of the maximum matching.

   Solution: Take a maximum matching and observe that the set of vertices of this matching (there are two times many vertices than edges in a matching) is a vertex cover for the graph. This is because if there is an edge not covered by these vertices, we can add it to the matching and get a bigger one. A disjoint collection of triangles shows that sometimes the minimum vertex cover is two times the size of the maximum matching.

5. In a bipartite graph $G$ where $V(G) = A \cup B$ we would like to match each vertex in $A$ with two distinct vertices in $B$ such that every vertex in $B$ is matched to at most one vertex in $A$. Formulate and prove a necessary and sufficient condition (in the spirit of Hall's theorem) for this to be possible.

   Solution: The condition is that any for any $X \subset A$ we have $|N(X)| \geq 2|X|$. To see this one can add another copy of every vertex in $A$ and connect it to the same set of vertices in $B$ as its twin vertex. Then the original Hall's condition is satisfied for the new graph once our condition is satisfied for the original graph. Finding a matching in the new graph for every vertex in $A$ and its copy is equivalent to matching in the original graph every vertex in $A$ with two vertices in $B$.