Graph Theory. Rom Pinchasi EPFL Fall-2023

December 27, 2023

Week 1: basic definitions and bi-partite graphs

- Define the notion of a simple graph. $G(V,E)$. Define what it means for two vertices to be neighbors. Degree of a vertex. Vertex that is incident to an edge.
- maximum number of edges in a graph on n vertices is $\binom{n}{2}$
- Examples: Empty graph, complete graph, a cycle, a path. A complete bipartite graph $G = (A \cup B, E)$.
- Theorem: $\sum_{v \in V} d(v) = 2|E|$. In particular, the sum of the degrees is even.
- Prove that in any graph with an odd number of vertices there is a vertex of even degree
- Let P be a set of 2023 light bulbs in the plane. Each has a switch. For every bulb we press its switch and also all the switches of the light bulbs at distance < 1 from it. Show that at the end there must be at least one light bulb that is ON.
- Define a bipartite graph.
- Not every graph is bi-partite. Examples: odd cycle.
- A graph is bi-partite iff it does not contain an odd cycle (mention what is a SUBGRAPH).

  Proof. One direction is clear. Assume now that G does not contain an odd cycle. Start with a vertex and color it red. Then color the nighbors Blue and continue. We cannot get a conflict. Finally, we claim that the coloring is a good coloring of the graph in two colors.

  We work on every connected component independently.
- Theorem: Every graph G contains a subgraph that has at least half of the edges of G.

Week 2: The adjacency matrix of a graph

- Define $A(G)$ with examples.

- What can we do with the adjacency matrix: A^k gives the number of ways to get from a vertex u to a vertex v with a tour of k edges in the graph.

- Give a small example with a graph on 4 vertices (path?).

- example: In how many ways we can get from u to v in k steps on the complete graph? NOT VERY SIMPLE. This is equivalent to writing a sequence of numbers from among $\{1,2,,n\}$ that starts with 1 and ends with 2 and every two adjacent elements are different.

- What is $A(G)^k$ for the complete graph? There is more than one way to do it. Notice: $A(G)^k = (J-I)^k$, where J is the $n \times n$ matrix all of whose entries are equal to 1. This gives a matrix that is equal to $\frac{1}{n}((n-1)^k - (-1)^k)$ off the main diagonal and $\frac{1}{n}((n-1)^k - (-1)^k) + (-1)^k$ on the main diagonal.

- In general we will need to compute $A(G)^k$. This is best done by diagonalizing $A(G)$. Show how.

- Easier for the complete bipartite graph on a and b vertices. DID NOT DO IN CLASS - LEFT FOR TA session.

- $(A+I)^k$ gives the number of ways to get from u to v in k steps where we can choose to stay in place if we want. DID NOT DO IT IN CLASS.

- Puzzle: We are given a graph G that could be any graph. In each vertex there is a light bulb and a switch. Pressing the switch changes the condition of the light bulb of at the vertex and its neighbors. Show that it is always possible to press some switches and light all the light bulbs. Example: A cycle, a path. the cube.

- Try an example.

- For the first solution we introduce the adjaceny matrix of a graph. $A(G)$. We want to find a vector v such that $(A(G)+I_n)v = (1,1,\ldots,1)$. We need to show that $(1,\ldots,1)$ lies in the span of the columns of $A(G)+I_n$. From linear algebra this is the same as showing that any vector that is orthogonal to the columns of $A(G)+I_n$ is also orthogonal to $(1,\ldots,1)$.

  Assume therefore that $u(A(G)+I_n) = 0$. Then $u(A(G)+I_n)u = 0$. But $u(A(G)+I_n)u = \langle u,(1,\ldots,1)\rangle$.

  WE DID NOT DO IN CLASS WHAT FOLLOWS:

- Solution by induction: For every vertex v consider $G-v$. Either we are done, or we know how to turn on all the vertices but v.

  If there is v with even degree, then we can turn ON all the vertices but v and its neighbors. Then we press the switch of v.

  We conclude that every vertex has odd degree. In particular, the number of vertices must be even. Now we can use the induction hypothesis for every $G-v$ and all of them in total light all the vertices.

Week 3: Connectivity and trees

- Define a connected graph. Give examples. Define a connected component. Show that a graph with n vertices and e edges has at least $n-e$ connected components.
- Define a tree: A connected graph with no cycles.
- Every tree has a leaf. In fact, at least two (unless $n=1$).
- Every tree on n vertices has $n-1$ edges. Proof: Add the edges one by one. After $n-1$ edges G has only one connected component.
- Theorem: G is a tree iff G is connected and has n vertices and $n-1$ edges.
- Theorem G is a tree iff G has no cycles and it has $n-1$ edges. Proof: Add edges one by one.
- In a tree there is a unique path between two vertices. Proof: By induction: If the first edge of the two paths is the same, remove it and conclude by induction. Otherwise, go along the first path until the first time you hit a vertex of the second path. This is a cycle.
- Theorem: Every connected graph has a SPANNING TREE. Proof: As long as there is a cycle remove edges.
- How many different spanning trees does the complete graph on $n \geq 2$ labeled vertices have? Answer: n^{n-2}. Cayley's formula

  Give some simple examples: 4 vertices.

  Proof: Prüfer sequences. Give labels to the vertices. Then Take the smallest leaf and write its neighbor. Continue like this until two vertices are left (connected). We get a list of $n-2$ numbers. Each between 1 and n. Every number appears one time less than its degree (check: sum of degrees is $2n-2$ and so we have a list of size $2n-2-n=n-2$). No two different trees have the same list because we can reconstruct the tree from the list. The reason is that at every moment we know who are the leaves of the remaining tree.

Week 4: More on trees: Graphs and Matrices

- remind Cayley's theorem and Prüfer encoding. Give and example and show why the graphs reconstructed from a sequence do not have cycles.
- example: How many are there with 5 vertices of degree 7, 3 vertices of degree 4 and the rest of the vertices being leaves?
- We define two important matrices for a graph:

  1. The vertex-edge incidence matrix. This is a $|V| \times |E|$ matrix. We will use an oriented variant: $E_{v,e} = 1$ if v is the smaller vertex of e. Otherwise $E_{v,e} = -1$.

  3. The laplacian $L(G) = D(G) - A(G) = EE^t$, where $D(G)$ is a diagonal matrix that has a degrees of the vertices of G on the main diagonal.

- Notice that the Laplacian matrix is singular. It has a 0 eigenvalue with eigen vector $(1,1,\dots,1)$.

- Theorem: The number of spanning trees of G is equal to $\det L(G)[i]$ for every $1 \le i \le n$.

- Theorem (Kirchhoff's matrix-tree theorem - will be proved next week). The number of spanning trees of a graph G is equal to $\det L(G)[i]$ for every i.

- Corollary: Theorem (Kirchhoff's matrix-tree theorem). The number of spanning trees of a graph G is equal to $\frac{1}{n}\lambda_1 \cdots \lambda_{n-1}$, where $\lambda_1,\dots,\lambda_{n-1}$ are all the non-zero eigen values of $L(G)$.

- Proof of the theorem: On one hand the linear term in the characteristic polynomial of $L(G)$ is $(-1)^{n-1}x\pi_{i=2}^{n}\lambda_i$. On the other hand the characteristic polynomial is $\det xI_n - L(G)$. By expanding the determinant we see that the linear term is equal to $\sum_{i=1}^{n}(-1)^{n-1}\det L(G)[i]$. This is equal to $n(-1)^{n-1}f(G)$.

- Example: When G is the complete graph, $L = nI_n - J$. This matrix has $n-1$ eigen values being equal to n. Therefore, $\frac{1}{n}\lambda_1 \cdots \lambda_{n-1} = n^{n-2}$.

- Example: How many spanning trees does the complete bipartite graph $K_{n,n}$ have?

- Solution: $L(G) = nI_{2n} - A(G)$. $A(G)$ has al its eigen values being equal to 0 except for two. They are equal to n and $-n$ with eigenvectors $(1,\dots,1)$ and $(1,\dots,1,-1,\dots,-1)$. Therefore, the nonzero eigenvectors of $L(G)$ are n with multiplicity $2n-2$ and $2n$ with multiplicity 1. Therefore, the number of spanning trees of G is n^{2n-2}.

Week 5: a little more on trees

- Cauchy-Binnet Formula: Let A and B be two $k \times m$ matrices, where $k \le m$. Then $\det AB^t$ is equal to $\sum_S \det A_S \det B_S$, where the sum is over all subsets S of k columns out of m.

- Proof of Cauchy-Binnet (DID NOT DO IN CLASS): Consider all the multi-linear and anti-symmetric functions on k vectors in $\mathbb{R}^m$. Observe that this is a linear space of dimension $\binom{m}{k}$. It is spanned by all the determinants $\det A_S$.

  Now consider $\det AB^t$ and notice that this is a multi-linear anti-symmetric fucntion of A. Therefore, it is equal to $\sum_{b_S \det A_S}$. Taking A to be I_k on S, we see that $b_S = \det B_S$.

- Proof of matrix-tree Theorem: We notice that any spanning tree corresponds to an $n-1$ by $n-1$ submatrix of E minus the first row (or any row) having a non-zero (therefore either $+1$ or -1) determinant. Now, it follows from Couchy-Binnet that the number of spanning trees is equal to the determinant of $E'E'^t$, where E' is E with the first row removed. This is precisely the determinant of $L(G)[i]$.

- Define a subgraph + some examples and explain the idea behind the theory of extremal graphs.
- How many edges can G have if it does not contain a triangle? Turan's theorem: at most $n^2/4$.
- Proof. Consider the vertex of highest degree. Connect each of its non-neighbors to its neighbors, thus increasing the number of edges. We obtain a bipartite graph. This finishes the proof.
- Same proof (with induction) for the case of forbidden K_r. The answer $\binom{r-1}{2}n^2/(r-1)^2 = \frac{n^2}{2}(1-\frac{1}{r-1})$

  Proof. Take the vertex with the highest degree. Among its neighbors there is no K_{r-1}. It follows that G has not more edges, than $(r-1)$-partite graph.
- Define in genral $Ex(H,n)$.
- How many edges can G have if it does not contain H and H is not bipartite (contains an odd cycle)? At least $n^2/4$. In this sense this is not very interesting. What is H is bi-partite?
- If G does not contain $K_{2,2} = C_4$, then G has $O(n^{3/2})$ edges.
- Example: Given n points and n lines in the plane. At most how many incidences are there?
- example: What is the maximum number of unit distances among n points in the plane? $n^{3/2}$ follows. Still Open.

Week 6: extremal graph theory

- Proof of the theorem: we count V's in G. On one hand at most n^2. On the other hand precisely $\sum_{i=1}^{n} d_i^2$. We use cauchy-Schwartz inequality: $(\sum d_i) \leq \sqrt{n}\sqrt{\sum d_i^2}$.

  (For a proof, consider the vectors $(1,\dots,1)$ and $(d(v_1),\dots,d(v_n))$ and use $\langle u,v\rangle \leq |u||v|$.)
- Show tightness:

  Consider $\mathbb{F}_2^2$. Define the incidence graph between the p^2 points and the p^2 lines. It has p^3 edges and no C_4.
- Same result for $K_{2,s}$ with dependency on s as a constant.
- Theorem (Kovari-Sos-Turan): If G does not contain $K_{r,s}$ for $r \leq s$, then G has $O(n^{2-\frac{1}{r}})$ edges. Similar proof using the inequality $\frac{1}{n}\sum d_i \leq \sqrt[r]{(\frac{1}{n}\sum d_i^r)}$.
- Let T be any tree (or forest). $Ex(T,n) = O(n)$.

  Proof: Let k be the number of vertices in T. Therefore k is also an upperbound on the degree of a vertex in T. We prove the theorem by induction on the number of vertices in T. Delete k edges incident to each vertex in G. G remains with at least $|E| - nk$ edges. By induction hypothesis G contains the tree T' which is T minus a leaf. Now just add the leat to the copy of T' in G. This proof gives $Ex(T,n) \leq ck^2n$, where k is the number of vertices of T.

- Theorem: Suppose G has average degree d, then there is a subgraph of G with avrage degree at least d and with minimum degree at least $d/2$.

  Proof: By induction on n. Assume there is a vertex of degree smaller than $d/2$. Remove this vertex and observe that the average degree in the remaining graph is at least $2(nd/2 - d/2)/(n-1) = d$. Conclude by induction.

- Example: A different proof showing that $Ex(C_4, n) \leq 100n^{3/2}$.

  Proof: Let G be a graph with more than $100n^{3/2}$ edges and no C_4. We may assume that the degree of every vertex of G is at least $100n^{1/2}$, or else we find such a subgraph G' of G. (equivalently, conclude by induction).

  We may assume that G is bipartite and the degree of each vertex is at least $50n^{1/2}$.

  Take any vertex x. It has at least $50n^{1/2}$ neighbors. They have at least 50^2n neighbors that are all different. A contradiction.

Week 7: Bondy-Simonovich, chromatic number and coloring

- Theorem: (Bondy-Simonovich): If G does not contain C_{2k}, the G has at most $1000(n^{1+\frac{1}{k}})$ edges.

  Proof: We prove the case $k = 3$ (the case $k = 2$ we already did!). The proof will give an idea of the proof for general k.

- Proof of the case of C_6. First we find a large bi-partite subgraph of G. Then we find a subgraph of G that is bi-partite and such that the degree of each vertex is at leasts $1000n^{1/3}$.

  We consider the neighbors of x that we denote by A and their neighbors that we denote by B we observe that the bipartite graph with parts A and B does not contain a path of length 5. Hence the number of edges in this graph is linear in the number of vertices. Therefore $200(|B| + |A|) \geq 1000|A|n^{1/3}$.

  Therefore, $|B| \geq 4|A|n^{1/3} \geq 4000n^{2/3}$. Also in particular, $|A| \leq n^{2/3}$. Next we denote the neighbors of B not in A by C. Again we observe (more tricky) that the bipartite graph with sides B and C cannot contain a path of length 5. Hence $200(|B| + |C|) \geq 4000n^{2/3}1000n^{1/3} - 200(|B| + n^{2/3})$.

  Hence, $400(|B| + |C|) \geq 4000n1000 - 200(n + n^{2/3})$, a contradiction.

- Define Chromatic number $\chi(G)$ of a graph. Examples: bi-partite graphs, complete graphs. Odd cycle. $\alpha(G)$, $\omega(G)$, $\delta(G), \Delta(G)$. Clique, anti-clique

- $\chi(G) \leq \Delta(G) + 1$.

- Clearly $\chi(G) \geq \omega(G)$. There are graphs with $\omega(G) = 2$ (no triangles) but $\xi(G)$ arbitrarily large.

  Proof: We can take $\chi(G)$ copies of G and for every transveral we add a vertex connected to each vertex in the transversal.

- example: The chromatic number of the plane: at least 4 at most 7. Very recently improved to between 5 and 7. In space between 6 and 15.

- Mention the four-color-theorem

- In how many ways can we legally color a graph G with k colors?

  The answer is given by the Chromatic Polynomial of the graph.

  The answer must be a polynomial by using the inclusion exclusion formula. For every edge e let A_e be the number of coloring with k colors such that the two vertices of e get the same color. Then we would like to estimate $|\cup_e A_e|$. By using the inclusion exclusion formula we see that this is a polynomial in k.

- Give some examples of the chromatic polynomial (complete graph, complete bipartite graph, empty graph).

Week 8: planar graphs

- define a planar graph and a planar map. vertices, edges, faces, unbounded face.

- examples: touching graphs of discs, countries. Trees, cycles.

- Mention Koebe theorem.

- Euler's formula $V - E + F = 1 + c$.

- conclude: a planar graph ($n \geq 3$) (with no parallel edges) has at most $3n - 6$ edges. This is best possible.

- Example: $K_{3,3}$ and K_5 are not planar.

- Mention Koratowski's Thm.

- Show that the same is true on the sphere. Talk about the polar mapping.

- corollary: every planar graph has a vertex of degree at most 5.

- Every planar graph can be colored with 6 colors.

- A bipartite planar graph contains at most $2n - 4$ edges.

- example: n disjoint red discs and n disjoint blue discs. How many pairs of red-blue discs intersect?

- DO NOT DO THIS IN CLASS!!! LEAVE FOR TA. A planar map is bipartite iff every face has an even size.

  Proof: one direction is clear. For the other direction, assume there is an odd cycle. Count the edges inside the cycle and on its boundary.

- example: a geometric graph with no pair of avoiding edges has at most $2n-2$ edges.

Week 9: crossings

- Reminder: a planar graph ($n \geq 3$) with no parallel edges has at most $3n-6$ edges. This is best possible.

- Example: $K_{3,3}$ and K_5 are not planar.

- Mention Kuratowski's Thm.

- A graph with $e > 3n-6$ edges must contain at least $e-3n$ crossings.

- Define the crossing number of a graph $Cr(G)$.

- Thm: if $e > 4n$, then $Cr(G) \geq \frac{1}{64}E^3/n^2$.

  Proof: Take every vertex with probability p and consider the resulting graph. In average $E(v^*) = pv$ $E(e^*) = p^2e$. $E(Cr^*) \leq p^4 Cr(G)$ We get $p^4 Cr(G) \geq p^2e - 3pn$. Therefore, $Cr(G) \geq e/p^2 - 3n/p^3$. Take $p = 4n/e$ we get $Cr(G) \geq \frac{1}{16}e^3/n^2 - (3/4)(1/16)e^3/n^2 = (1/64)e^3/n^2$.

- example: number of incidences between m lines and n points in the plane.

- example: number of unit distances among n points in the plane.

- example: if $e = cn^2$ then $Cr = c'n^4$.

- example: Elekes' theorem: Let A be a set of numbers. Then either $|A+A|$ or $|A \cdot A|$ is at least as large as $|A|^{5/4}$.

  Proof. Consider the points $\{(a+b, cd) \mid a,b,c,d \in A\}$ and consider the set of lines $\{y = a(x-b) \mid a,b \in A\}$.

  We have $n = |A+A||A \cdot A|$ points and $m = |A|^2$ lines. Observe that every line is incident to at least $|A|$ points namely $(x+b, ax)$ for every $x \in A$. Therefore the number of incidences is greater than $|A|^3$. On the other hand this is at most $m^{2/3}n^{2/3} + n + m$. We get $n \geq |A|^{9/2}/|A|^2 = |A|^{5/2}$.

Week 10: matchings in bipartite graphs

- Define a matching in a graph. Define a perfect matching.

- Hall's theorem: A bipartite graph $G = A \cup B$ has a matching of all the vertices in A iff for every $X \subset A$ we have $|X| \leq |N(X)|$.

  Proof: One direction is clear. For the other direction we prove the theorem by induction on $|A|$. Let $A' \subset A$ be such that $0 < |A'| < |A|$ and $|N(A')| = |A'|$. If there is no such A' we are done by matching an arbitrary vertex of A and considering the remaining graph.

  By induction there is a matching for all the vertices in A'. Consider the graph between $A \setminus A'$ and $B \setminus N(A')$. It satisfies Hall's condition. And we are done again by induction.

- Notice that if $|A| = |B|$, then we have a perfect matching in Hall's theorem.
- example: Define a bi-stochastic matrix. Then in every bi-stochastic matrix one can find a permutation with all entries positive.
- The puzzle with the keys and locks.
- Proof of Hall's theorem via augmenting paths: Let M be a maximum size matching in G. Assume to the contrary that $x \in A$ is not covered in M. We consider all "alternating" paths starting at x. We define A' to be the set of all vertices in A that are the endpoints of such alternating path. We let B' to be the set of vertices in B that are the endpoints of such paths. We observe that every vertex in B' is in M or else we have an augmenting path for M. We also observe that every vertex in A' (other than x) is in M. Therefore, $|A'| = |B'|$. We observe that $N(A' \cup \{x\}) = B'$. We thus get a contradiction to Hall's condition.

Week 11: more on matchings in bipartite graphs

- Define an augmenting path for a matching M. Notice that also an edge that is independent of M is an augmenting path.
- Thm. A matching is maximal in size iff it does not have an augmenting path.

  Proof. One direction is clear. In the other direction assume M is not maximal and consider $M \cup M^*$, where M^* is a larger matching. It is a union of cycles and alternating paths (one edge in M and one in M^*). There must be a path with more edges of M^*. This is an augmenting path for M.
- puzzle: Archipeleg puzzle. You have odd number of islands you take a flight to one and then use the boats lines. The player who cannot make a move to a NEW island loses. Show that the first player can always win.
- Example: We are given $n \times n$ table with the numbers $1, \dots, n$ each appearing n times. Then we can permute the columns such that in every row we have all the numbers $1, \dots, n$.
- Let G be a bi-partite graph: $V(G) = A \cup B$. Assume that for every $X \subset A$ we have $N(X) \geq |X| - 100$. Prove that there is a matching in G that involves all the vertices of A, except maybe for at most 100 vertices.
- Define a vertex cover of a graph. Give some examples. Talk about minimum vertex cover of a graph.
- Konig's Thm. In a bipartite graph the size of the maximum matching is equal to the size of the minimum vertex cover.

  Proof. One direction is clear. For the other direction we need to show that the maximum matching is at least the size of the minimum vertex cover.

  Assume that the maximum matching has size s. Then there must be a set X such that $N(X) \leq |X| - (|A| - s)$. But then we can take $N(X) \cup (A \setminus X)$ as a vertex cover. The size of this set is $\leq s$.

Week 12: Ramsey Theory

- Explain the idea in short. Show $R(3,3)=6$.
- Ramsey's Theorem: $R(s,t) \leq \binom{s+t-2}{s-1}$.
- Recall $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$.
- By induction. Observe that $R(s,t) \leq R(s-1,t)+R(s,t-1)$. Basis of induction $R(s,2)=s$ and $R(2,t)=t$.
- In particular: $R(s,s) \leq \binom{2s-2}{s-1} \leq 2^{2s-2}$.
- Show $R(s,s,) \leq 2^{2s}$ directly.
- This implies the infinite Ramsey theorem. State and prove.
- example: every sequence of reals contains a monotone subsequence. In particular, every bounded sequence has a converging subsequence (Bolzano-Weirstrass)
- There are colorings of the edges of the complete graph on 2^s vertices with no monochromatic K_s.

  Proof: We color randomly every edge by red or blue independently with probability $\frac{1}{2}$. The probability of a given set of s vertices to form a monochromatic complete graph is $\frac{2}{2^{\binom{s}{2}}}$. The number of different s-sets of vertices is $\binom{n}{s}$. Therefore, if $\frac{2}{2^{\binom{s}{2}}}\binom{n}{s} < 1$, then there is a coloring of the edges where no set of s vertices form a monochromatic compltete subgraph.

  Notice that $\binom{n}{s} \leq n^s$. Therefore, it will be enough if $n^s \leq 2^{\binom{s}{2}}$. Hence, $n \leq 2^{s/2}$ is good enough. This shows that $R(s,s) \geq 2^{s/2}$.

Week 13: more on Ramsey Theory

- Generalization to k colors. Define $R_k(s_1,\ldots,s_k)$.
- Theorem $R_k(s_1,\ldots,s_k)$ is finite.

  For the proof we use induction: $R_{2k}(s_1,\ldots,s_{2k}) \leq R_k(R_2(s_1,s_2),\ldots,R_2(s_{2k-1},s_{2k}))$. Also $R_k(s_1,\ldots,s_k) \leq R_{k-1}(R_2(s_1,s_2),s_3,\ldots,s_k)$.
- Schur's Theorem: For every k there is $n(k)$ such that for any $n > n(k)$ and any coloring of the number $1,2,\ldots,n$ with k colors one can find x,y,z of the same color such that $x+y=z$.
- Proof: We take $n(k) = R_k(3,3,\ldots,3)$. Then given a coloring c for the numbers $1,2,\ldots,n$ we color the edges of K_n by $c(x,y) = c(|x-y|)$. We find a monochromatic triangle with vertices $x<y<z$. Then $(y-x)+(z-y)=(z-x)$ and all have the same color. Notice that it is possible that $y-x=z-y$.
- Application to Fermat's last theorem. Remind the theorem. $x^n+y^n=z^n$ has no positive solution for $n>2$.

- Theorem $x^n + y^n + z^n$ has solution modulo p for any large enough (in terms of n) p.

  Proof. The multiplicative group Z_p^* is cyclic. This means that there is a such that the powers of a produce all the number $1, 2, \ldots, p-1$. Color every number a^s (modulo p) by $s \mod n$. Then, by Schur's theorem, we can find x, y, z such that $x + y = z$ and they all have the same color. That is $a^{ni+s} + a^{nj+s} = a^{nk+s}$. Then $(a^i)^n + (a^j)^n = (a^k)^n$.

  IMPORTANT: the numbers a^s are taken modulo p, or else they are too big in the range where not every number is a power of a.

- Generalization to hypergraphs. Define 3-uniform hypergraphs.

- Define $R^3(k,l)$. Show that $R^3(4,4)$ is finite. hint: $R^3(4,4) \leq 1 + R(4,4)$.

- More generally, show that $R_3(s,t)$ is finite.

  Observe: $R^3(s,t) \leq 1 + R_2(R^3(s-1,t), R^3(s,t-1))$.

- and so on...

- example: given a set of n points in the plane one can find there a subset of large size in convex position.

  Proof: Color the triples in two colors. We actually find a large convex chain or a large concave chain,

- Every set of n points in the plane either have many points in general position, or it has many points on one line.

- Remark: many times the bounds given by Ramsey's theory to geometric problems are far from being tight. There is a good reason for this.

Week 14: Euler and Hamilton cycles and paths

- Define an Euler tour + examples.

- Thm. G has Euler tour iff G is connected and the degree of each vertex is even.

  Proof: consider maximal walk (necessarily closed).

- define an Euler path.

- Thm. G has an Euler path iff it is connected and all the degrees are even, or all but two.

- Define a directed graph.

- Thm: a directed graph has an Euler tour iff G is connected and every vertex has the same in-degree as out-degree.

  a circular sequence of bits of length 2^{100} where all the contiguous 2^{100} possibilities appear (each one once).

- Define a Hamilton cycle.n give some examples.
- Theorem (Dirac) if $n \geq 3$ and $\delta(G) \geq n/2$, then G contains a Hamilton cycle.

  Proof. First notice that G is connected. Consdier a maximal path $v_1 v_2 \ldots v_k$. All the neighbors of v_1 and v_k must be in the path. Because of the condition on $\delta(G)$ there must be i such that v_{i+1} is a neighbor of v_1 and v_i is a neighbor of v_k. We get a cycle of length k. This must include all vertices, or else we can get a larger path by using an edge that goes out of the cycle.
- Show the result is best possible by taking two copies of K_m with a common vertex.
- Notice: it is enough to assume in the proof that $d(u) + d(v) \geq n$ for every non-adjacent vertices u and v.