Graph Theory 2023 (EPFL): Problem set of week 11

November 28, 2023

1. At EPFL 100 students stay over Christmas on campus. Santa Claus brings over 10,000 presents for the 100 students. The presents are of 100 types where of each type there are 100 units. Santa Claus wanted each student to take one present of each type but the students at EPFL drank too much wine that night and each of them took randomly 100 of the presents. Next morning they realized their mistake and wanted to fix it. Santa Claus allowed them to switch presents among themselves (any two students can switch presents with one another, you give k presents and get k presents) but only in such a way that if you get a present from someone it remains yours forever and you cannot switch it anymore.

   Show that if the students are very clever (and they are because they are EPFL students) they can switch the presents in such a way that at the end each student has all the 100 types of presents.

   Hint: It is not such an easy puzzle but it has a one line solution if you recall one of the puzzles we did in class today.

2. In a 10×10 matrix all the numbers from 1 to 100 are written in an arbitrary way. At every step we are allowed to permute all the rows, or all the columns. Our goal is to reach the situation where the numbers are in order from left to right and from top to bottom: the first row is $1, 2 \ldots, 10$. The second row is $11, 12, \ldots, 20$, and so on. What is the minimum number k of steps that can guarantee that we can always reach our goal in at most k steps?

3. Let G be a bipartite graph that is k-regular. That is, the degree of every vertex in G is equal to k. Prove that the edges in G can be partitioned into k perfect matchings.

4. Show that if G any graph (not necessarily bipartite, then the size of the minimum vertex cover is at most two times the size of the maximum matching. Find examples with arbitrary number of vertices where it is equal to two times the size of the maximum matching.

5. In a bipartite graph G where $V(G) = A \cup B$ we would like to match each vertex in A with two distinct vertices in B such that every vertex in B is matched to at most one vertex in A. Formulate and prove a necessary and sufficient condition (in the spirit of Hall's theorem) for this to be possible.