Graph Theory 2023 (EPFL): Problem set of week 10

November 23, 2023

1. If $G = A \cup B$ is bipartite and the degree of every vertex in A is $\geq k$ and the degree of every vertex in B is $\leq k$, then G has a matching of all the vertices in A.

   Solution: We check that Hall's condition is satisfied. Take $X \subset A$ and consider $N(X)$. The number of edges running between X and $N(X)$ is $\geq k|X|$ (because the degree of every vertex in A is at least k). On the other hand it is at most $k|N(X)|$ (because the degree of every vertex in B is at most k). It follows now that $k|X| \leq k|N(X)|$, or in other words $|X| \leq |N(X)|$, as desired.

2. Let G be a bipartite graph: $V(G) = A \cup B$. Assume that there is a matching M_A in G that includes all the vertices in $A' \subset A$ and there is another matching M_B in G that includes all the vertices in $B' \subset B$. Show that there is a matching M in G that includes all the vertices in $A' \cup B'$.

   Is the same true for general graphs?

   Solution: Consider the union of M_A and M_B. This is a graph where the degree of each vertex is at most 2. Therefore, it is a union of paths and cycles. Notice that the cycles must be even cycles.

   Consider any one of the paths and observe that if it has an even number of edges, then it must have its end vertices either both in A or both in B (here we use the fact that G is bipartite). However, if say both end vertices of a path of even length are, say, in A, then it cannot be that both are also in A' because then this path should have ended on both ends by edges in M_A, which is impossible for a path of even length.

Now it should be easy to get M. From each (even) cycle take half of the edges to M (every second one). From every path with $2k+1$ (odd number of) edges take $k+1$ edges and all vertices to M. From paths of even length take half of the edges to M and make sure that the vertex that is not included is the one not in A' (respectively, not in B').

This will not work if the graph G is not bipartite. For example think of G being a triangle and let two vertices be in A' while the third being in B'.

3. Let G be a graph and let M be a matching in G. Show that there is a maximum matching in G (that is, matching of maximum possible size) that involves every vertex in the matching M.

   Solution: Consider the maximum matching M' of G that uses the maximum number of vertices in the matching M. If all vertices of M are used, we are done. Otherwise, take a vertex x_1 in M that is not in M'. Let x_2 be the match of x_1 in M. x_2 must be in M', or else we could add (x_1, x_2) to M' contradicting maximality. Now let x_3 be the match of x_2 in M' (there must be such x_3, why?). The vertex x_3 must be in M, or else we could take for M' the edge (x_1, x_2) instead of (x_2, x_3) and get a maximum matching sharing more vertices in M. Let x_4 be the match of x_3 in M. Again, x_4 must be in M', or else $x_1x_2x_3x_4$ is an augmenting path for M'. Let x_5 be the match of x_4 in M'. We claim that x_5 must be in M. Otherwise, replace x_2x_3 and x_4x_5 in M' with x_1x_2 and x_3x_4 and get a maximum size matching with more common vertices with M. We continue like this and either we get an augmenting path for M', which is impossible, or we can replace M' by another maximum matching with more common vertices with M, or we can continue forever which is impossible because we see more and more vertices and our graph is finite. This contradiction shows that M' contains every vertex in M.

4. Let A be an $n \times n$ bi-stochastic matrix. Show that one can find permutation matrices $P_1, \ldots, P_k$ and $0 \leq \lambda_1, \ldots, \lambda_k \leq 1$ such that $A = \sum_{i=1}^{k} \lambda_i P_i$. (Recall that a permutation matrix is a matrix that has precisely one 1 entry in each row and each column and all the other entries are equal to 0.)

   Solution: By the theorem we have seen in class, there is a permutation π such that $a_{i,\pi(i)} > 0$ for every i. Let $\lambda = \min_i(a_{i,\pi(i)})$. Let P be the permutation matrix that has 1's at the entries $(i, \pi(i))$ for $i = 1, \ldots, n$.

If $\lambda = 1$, then $A = P$, and we are done. Otherwise, $\frac{1}{1-\lambda}(A - \lambda P)$ is again a nonnegative bi-stochastic matrix (why?) and it has at least one more entry than A that is equal to 0. We can therefore conclude by the induction hypothesis (on the number of nonzero entries in A) that $\frac{1}{1-\lambda}(A - \lambda P) = \sum_{i=2}^{k} \lambda_i P_i$, where $\sum_{i=2}^{k} \lambda_i = 1$. It now follows that $A = \lambda P + (1 - \lambda) \sum_{i=2}^{k} \lambda_i P_i$ and this is the desired result (check!).