Graph Theory 2023 (EPFL): Problem set of week 10

November 23, 2023

1. If $G = A \cup B$ is bipartite and the degree of every vertex in A is $\geq k$ and the degree of every vertex in B is $\leq k$, then G has a matching of all the vertices in A.

2. Let G be a bipartite graph: $V(G) = A \cup B$. Assume that there is a matching M_A in G that includes all the vertices in $A' \subset A$ and there is another matching M_B in G that includes all the vertices in $B' \subset B$. Show that there is a matching M in G that includes all the vertices in $A' \cup B'$.

   Is the same true for general graphs?

3. Let G be a graph and let M be a matching in G. Show that there is a maximum matching in G (that is, matching of maximum possible size) that involves every vertex in the matching M.

4. Let A be an $n \times n$ bi-stochastic matrix. Show that one can find permutation matrices $P_1, \ldots, P_k$ and $0 \leq \lambda_1, \ldots, \lambda_k \leq 1$ such that $A = \sum_{i=1}^{k} \lambda_i P_i$. (Recall that a permutation matrix is a matrix that has precisely one 1 entry in each row and each column and all the other entries are equal to 0.)