Graph Theory 2023 (EPFL): Problem set of week 9

November 16, 2023

1. Given $n > 9$ points in the plane such that no three of them are on a line. Show that one can find at least cn^4 (for some absolute constant c) quadruples of points that form the set of vertices of a convex quadrilateral.

   Solution: We notice that any two crossing segments give rise to a unique quadruple of points forming a convex quadrilateral. Consider the complete geometric graph with the n points as vertices. It has $\binom{n}{2}$ edges and therefore, at least $\frac{1}{64}e^3/n^2$ crossings. Because $e = \binom{n}{2} > n^2/4$ we get at least cn^4 crossing and therefore at least cn^4 convex quadrilaterals.

2. Find geometric graphs with n vertices and e edges and only ce^3/n^2 crossings for every e and n.

   Hint: consider a set of n points on a unit circle. Then take the "shortest" possible chords.

   Solution: Take n vertices of a convex n-gon. Then take all the n edges between points of distance 1 (distance 1 is adjacent vertices on the boundary of the n-gon). Then take the n edges between vertices if distance 2 and so on. We do this until we get e edges. This means that we took all the chords of "length" (number of edges of the polygon between the two vertices) smaller than or equal to e/n.

   Now, an edge of "length" k crosses precisely $2(k-1)$ edges of every $i > k$. In this way we count all the crossings.

   Therefore, we have $\sum_{k=1}^{e/n} n \cdot 2(k-1) \cdot (e/n - k)$. This gives the desired result.

3. G is a graph with n vertices that can be drawn in the plane in such a way that every edge is crossed by at most one other edge. Show that the number of edges in G is at most $10n$.

   Solution: By our assumptions the number of crossings in the drawing of G is at most $e/2$. On the other hand we know that if the number of edges in G is at least $4n$ (if it is not, we are done), then the number of crossings must be at least $\frac{1}{64}e^3/n^2$. This implies

$$\frac{1}{64}e^3/n^2 \leq e/2.$$

   From here we get $e \leq \sqrt{32}n$.

4. Let G be a graph with n vertices and e edges. Prove that G has at least ce^3/n^2 distinct paths of three edges.

   Solution: There is more than one way to do this. We will adapt the probabilistic argument in the proof of the crossing lemma:

   We know that any graph with more than $30n$ edges contains a path of length 3. Therefore, the number of distinct paths of length 3 in a graph with n vertices and e edges is at least $e - 30n$.

   Denote by Z the number of distinct paths of length 3 in our graph. We take every vertex with probability p and consider the graph on the resulting set of vertices. The number of vertices in the new graph is in expectation pn. The number of edges is in expectation p^2e. The number of paths of length 3 is in expectation p^4Z.

   Therefore, $p^4Z \geq p^2e - 30pn$. We get $Z \geq e/p^2 - 30n/p^3$. Taking $p = cn/e$ for an appropriate constant c (that you should find) gives the desired result.