Graph Theory 2023 (EPFL): Problem set of week 8

November 10, 2023

1. Let P be a regular n-gon in the plane. Prove that it is not possible to draw more than $n-3$ diagonals of P without two of them crossing each other.

   Solution: Imagine that we draw k diagonals with no crossing. Then we can consider the resulting picture as a planar map. In addition we can add an extra vertex outside of the P and connect it by non crossing edges to each of the vertices of P. We get a planar map with $n+1$ vertices, $2n+k$ edges. We know that $2n+k$ should be less than or equal to $3(n+1)-6$. This gives $k \leq n-3$.

2. Let G be a planar graph. Show that G is bi-partite if and only if there is a drawing of G in the plane where all the bounded faces are of even size (that is, quadrangles, hexagons, etc.).

   Solution: One direction is clear because if G is bipartite, then it has no odd cycle and in particular every face (in any drawing of G as a planar map) must have an even size.

   Assume now that G is drawn in the plane as a planar map with all the bounded faces having even size. To show that G is bipartite, assume to the contrary that it contains an odd cycle C. Follow the drawing of C in the plane and observe that the region bounded by C is a union of faces. When we sum up the sizes of these bounded faces (this sum must be even) we count every edge that is not on C, but surrounded by C, twice and every edge on C we count just once. It follows that the number of edges on C must be even.

3. Let G be a planar graph on n vertices that can be drawn as a planar map in the plane such that every face in this map has size at least k. Show that the number of edges in G is at most $\frac{k}{k-2}n - \frac{2k}{k-2}$.

Solution:

From Euler's formula $V - E + F \geq 2$. Because every face has size at least k then we have $2E \geq kF$ (by summing up the sizes of the faces and observing that every edge is counted twise). Substitute $F \leq (2/k)E$ in Euler's formula and get the desired result after elementary manipulation.

4. What are all the essentially different planar maps for which it is known that all the vertices have the same degree and all the faces have the same size?

   Solution: We know already that in any planar graph there is a vertex of degree 5 or smaller. Therefore, the possible dgerees are $2, 3, 4, 5$. If all vertices have degree 2, then the graph must be a cycle (it cannot be a union of cycles because then the faces will not have the same size).

   If the degrees are all 3, then $3V = 2E$. Together with Euler's formula $V - E + F = 2$ we get $3F - E = 6$. If all the faces are of size 3, then $2E = 3F$ and we get $E = 6$ and $F = 4$. This graph exists (draw it). It looks like a pyramid. If all the faces are of size 4, then $2E = 4F$ and with $3F - E = 6$ we get $F = 6$ and $E = 12$. Such a graph exists. It looks like the three dimensional cube. If all the faces have size 5, then $2E = 5F$. Together with $3F - E = 6$ we get $F = 12$ and $E = 30$. Such a graph exists. Try to draw it. If all the faces have size k where $k \geq 6$, then $2E = kF$. This contradicts $3F - E = 6$ because then $kF = 6F - 12$.

   Next we consider the case where all the degrees are 4 (we get one possible solution) and 5 (one possible solution).

5. Let P be a set of n points in the plane such that no three of them are on one line. For every two points draw the line segment connecting them. Show that if $n \geq 7$ (in fact even for $n \geq 5$), then there is always a point in the plane that belongs to precisely two of the segments.

   Solution: Consider the drawing off all the segments and put a vertex at every crossing point of segments. We get a planar map. It must have a vertex of degree 5 or less. It cannot be one of the original n points because their degree is $n - 1 \geq 6$. So it must be one of the crossing points. This crossing point must be a crossing pf two segments, or else its degree is at least 6.