Graph Theory 2023 (EPFL): Problem set of week 7

November 7, 2023

1. Let $A_1, \dots, A_n$ be n subsets of $\{1, \dots, n\}$. It is known that for every $i \neq j$ we have $|A_i \cap A_j| < 5$.

   Prove that $\sum_{i=1}^{n} |A_i| \leq 100n^{3/2}$.

   Solution: Define a bipartite graph between the subsets $A_1, \dots, A_n$ on one side and the numbers $1, 2, \dots, n$ on the other side. Connect a set to all the elements that belong to it. Notice that the number of edges is precisely $\sum_{i=1}^{n} |A_i|$. Notice also that this bipartite graph does not contain $K_{2,5}$, where the 2 are on the subsets side and the 5 are on the numbers side.

   By a problem from last week such a graph cannot contain more than $100n^{3/2}$ edges. (Or, equivalently, one can adjust the proof of Kovari-Sos-Turan theorem to this case).

2. Prove that if a graph G on n vertices does not have any cycle of length smaller than or equal to $2k$, then the number of edges in G is at most $10n^{1+\frac{1}{k}}$.

   Hint: This is much easier than the Bondy-Simonovich theorem. Assume that G has more than $10n^{1+\frac{1}{k}}$ edges. We may assume that the degree of every vertex in G is at least half of the average degree, as we have seen in class. Start from any vertex x in G and consider its neighbors and their neighbors...

   Solution: We follow the hint. Assume to the contrary that G has more than $10n^{1+\frac{1}{k}}$ edges. Then the average degree of a vertex in G is more than $20n^{1/k}$.

   Let G' be a subgraph of G with average degree more than $20n^{1/k}$ and minimum degree of at least $10n^{1/k}$. Of course also G' cannot contain any cycle of length smaller than or equal to $2k$. Take any vertex x of G'. Consider its neighbors (at least $10n^{1/k}$ of them). Then their neighbors (at least $100n^{2/k}$ because they are all "new" vertices, or else we have a short cycle). Continue like this k steps. We get a set of $10^k n$ new vertices which is impossible.

3. Let G be a graph on n vertices $v_1, \ldots v_n$. Assume that for every i, the vertex v_i has at most 10 neighbors from among $v_1, \ldots, v_{i-1}$ (but may have more neighbors in G). Prove that the chromatic number of G is at most 11.

   Solution: We assign the colors to $v_1, \ldots, v_n$ sequentially. When we assign the color of v_i we just need it to be different from all the colors of its neighbors from among $v_1, \ldots, v_{i-1}$. But there are at most 10 such neighbors and we can use 11 colors. Hence there is a color that we can use for v_i and continue to v_{i+1}. This way every two neighboring vertices must get different colors. If v_i and v_j are neighbors and $i < j$ (without loss of generality), then when we assigned the color for v_j we made sure it is different for the color of v_i.

4. In how many ways can we color a cycle of length 5 in 10 colors such that no two neighboring vertices get the same color?

   Solution: One way is to find the chromatic polynomial of C_5. We can directly do the calculation through the inclusion-exclusion formula:

   Let A_e be the set of colorings where both vertices of the edge e of C_5 get the same color (this is bad coloring). The answer is $10^5 - |\cup_e A_e|$. The cardinality of the union is computed through inclusion-exclusion formula.

   The answer is: $10^5 - \sum_a 10^4 + \sum_{a,b} 10^3 - \sum_{a,b,c} 10^2 + 10^1$. a, b, c represent edges in C_5. This is $10^5 - 10^4\binom{5}{1} + 10^3\binom{5}{2} - 10^2\binom{5}{3} + 10$. The coefficient of 10^1 is not $\binom{5}{4}$ because any choice of four edges from C_5 is essentially the same and implies that the color of all vertices is the same.