Graph Theory 2023 (EPFL): Problem set of week 7

November 1, 2023

1. Let $A_1, \dots, A_n$ be n subsets of $\{1, \dots, n\}$. It is known that for every $i \neq j$ we have $|A_i \cap A_j| < 5$.

   Prove that $\sum_{i=1}^{n} |A_i| \leq 100n^{3/2}$.

2. Prove that if a graph G on n vertices does not have any cycle of length smaller than or equal to $2k$, then the number of edges in G is at most $10n^{1+\frac{1}{k}}$.

   Hint: This is much easier than the Bondy-Simonovich theorem. Assume that G has more than $10n^{1+\frac{1}{k}}$ edges. We may assume that the degree of every vertex in G is at least half of the average degree, as we have seen in class. Start from any vertex x in G and consider its neighbors, and their neighbors...

3. Let G be a graph on n vertices $v_1, \dots v_n$. Assume that for every i, the vertex v_i has at most 10 neighbors from among $v_1, \dots, v_{i-1}$ (but may have more neighbors in G). Prove that the chromatic number of G is at most 11.

4. In how many ways can we color a cycle of length 5 in 10 colors such that no two neighboring vertices get the same color?