Graph Theory 2023 (EPFL): Problem set of week 4

October 16, 2023

1. How many trees on 22 vertices are there with 4 vertices of degree 3, 3 vertices of degree 5, and 15 leaves?

   Solution: We recall Prüfer encoding. We first need to choose the 4 vertices of degree 3. This can be done in $\binom{22}{4}$ ways. Then we choose the 3 vertices of degree 5. This can be done in $\binom{18}{3}$ ways. The rest of the vertices are leaves. Now we need to count the number of sequences of length 20 such that there are 4 elements each of which appears twice in the sequence and 3 elements each appearing 4 times in the sequence. This can be done in $\frac{20!}{2!^4 4!^3}$ ways. The answer is the product of all the three numbers:

   $$\binom{22}{4}\binom{18}{3}\frac{20!}{2!^4 4!^3}.$$

2. Let G be a graph with n vertices and n edges. Show that G has at most n different spanning trees. What is the minimum number of spanning trees for such a graph if it is known to be connected?

   Solution: It is clear that G has at most n spanning trees because a (spanning) tree has $n-1$ edges and there are at most $\binom{n}{n-1} = n$ ways to choose $n-1$ edges out of n edges.

   We claim that if G is connected, then it must have at least 3 spanning trees. To see this consider a spanning tree T of G. T consists of $n-1$ of the edges of G. Let e be the edge of G not in T. When we add e to T, it creates a cycle. A unique cycle. This cycle has length 3 or more. Removing any edge of G from this cycle will result in a spanning tree of G. Therefore, G must have at least 3 spanning trees.

3. Consider the graph G on the set of vertices $A \cup B \cup C$ such that $|A| = |B| = |C| = n$ and we connect two vertices by an edge if and only if they belong to two different sets from A, B, and C. How many spanning trees does G have?

   Solution: Notice that the degree of every vertex in G is equal to $2n$. Therefore, $L(G) = 2nI_{3n} - A(G)$. Then matrix $A(G)$ is quite simple and has degree 3 (there are only three types pf rows in $A(G)$). This means that all the eigenvalues of G are equal to 0 except for three. $A(G)$ has one eigenvalue that is equal to $2n$ with eigenvector that is all 1's. There are two more nonzero eigenvalues for $A(G)$: $-n$ and $-n$ corresponding for example to the eigenvectors v that is 0 on A, $+1$ on B, and -1 on C and also u that is 0 on B, $+1$ on A and -1 on C. Therefore, the nonzero eigenvalues of $L(G)$ are $2n = 2n - 0$ (multiplicity $3n - 3$), $0 = 2n - 2n$ with multiplicity 1 and $3n = 2n - (-n)$ (multiplicity 2). By the Matrix-Tree Theorem, the number of spanning trees of G is

$$\frac{1}{3n}(2n)^{3n-3}(3n)^2.$$

4. Let $G = K_{r,s}$ be the complete bi-partite graph on r and s vertices. That is, $V(G) = A \cup B$ such that $|A| = r$ and $|B| = s$. The edges of G are all the pairs of vertices where one is from A and the other is from B. How many different spanning trees does $K_{r,s}$ have?

   Hint: By considering the rank of $L(G) - rI_n$ deduce that $L(G)$ has many eigenvalues that are equal to r. How many? Do the same for s. We know also that one eigenvalues must be 0 and the remaining eigenvalue we can find by considering the trace of $L(G)$ that is the sum of all eigenvalues. (You may want to consider the case $r = s$ separately, if you wish.)

   Solution: We follow the hint. $L(G) - rI_n$ $(n = r + s)$ has rank at most $r+1$ because it has s identical rows. It follows that the eigenvalue r has multiplicity at least $s - 1$. Similarly, the eigenvalue s has multiplicity at least $r - 1$. There is one eigenvalue of $L(G)$ that is equal to 0 (this is always the case). The trace of $L(G)$ is equal to $2sr$. It follows that there is another eigenvalue that is equal to $2rs - r(s - 1) - s(r - 1)$, that is: $r + s$. The number of spanning trees of G is therefore,

$$\frac{1}{r+s}r^{s-1}s^{r-1}(r+s) = r^{s-1}s^{r-1}.$$

   Given the answer, can you now give a combinatorial proof for this result using a method similar to the Prüfer code?

5. Let G be a graph on n vertices. Assume G has precisely k connected components. Prove that the rank of $L(G)$ is equal to $n - k$.

   Solution: We notice that if G is connected, then the rank of $L(G)$ is equal to $n - 1$. This follows even from the matrix tree theorem itself because the principle minors of $L(G)$ are non-singular and so $L(G)$ has rank of at least $n - 1$. If G has k connected components, then $L(G)$ is a matrix that is composed of k square blocks arranged in a diagonal. One block for every component. Each block represent a connected graph and hence has rank equal to one less than its size. Therefore, the rank of $L(G)$ is $n - k$.