Graph Theory 2023 (EPFL): Problem set of week 4

October 12, 2023

1. How many trees on 22 vertices are there with 4 vertices of degree 3, 3 vertices of degree 5, and 15 leaves?

2. Let G be a graph with n vertices and n edges. Show that G has at most n different spanning trees. What is the minimum number of spanning trees for such a graph if it is known to be connected?

3. Consider the graph G on the set of vertices $A \cup B \cup C$ such that $|A| = |B| = |C| = n$ and we connect two vertices by an edge if and only if they belong to two different sets from A, B, and C. How many spanning trees does G have?

4. Let $G = K_{r,s}$ be the complete bi-partite graph on r and s vertices. That is, $V(G) = A \cup B$ such that $|A| = r$ and $|B| = s$. The edges of G are all the pairs of vertices where one is from A and the other is from B. How many different spanning trees does $K_{r,s}$ have?

   Hint: By considering the rank of $L(G) - rI_n$ deduce that $L(G)$ has many eigenvalues that are equal to r. How many? Do the same for s. We know also that one eigenvalue must be 0 and the remaining eigenvalue we can find by considering the trace of $L(G)$ that is the sum of all eignvalues. (You may want to consider the case $r = s$ separately, if you wish.)

5. Let G be a graph on n vertices. Assume G has precisely k connected components. Prove that the rank of $L(G)$ is equal to $n - k$.