Graph Theory 2023 (EPFL): Problem set of week 3

October 9, 2023

1. Show that if $d_1, \ldots, d_n$ are n natural numbers such that $\sum_{i=1}^{n} d_i = 2n-2$, then there is a tree T whose set of degrees is precisely $d_1, \ldots, d_n$.

   Solution: For example, by induction on n. For $n = 1$ this is trivial. For $n = 2$ we must have $d_1 = d_2 = 1$ and the tree on two vertices is an (the only) example. Assume therefore that $n > 2$. For the induction step, notice that we must have at least one (in fact at least two) d_1 that is equal to 1 for otherwise $\sum_{i=1}^{n} d_i \geq 2n$. We take out this d_1 and then we find some d_j such that $d_j > 1$. This is possible for otherwise $\sum_{i=1}^{n} d_i = n$ this is smaller than $2n - 2$ if $n > 2$. We replace d_j with $d_j - 1$ and find a corresponding tree by the induction hypothesis. Then we add one leaf to v_j and get the desired tree on n vertices.

2. a) Let T be a tree and let e be an edge not in T. Show that if we add e to T we get a graph with precisely one cycle.

   Solution: We clearly get a cycle because we saw in class that every graph on n vertices and more than $n - 1$ edges must have a cycle. Assume to the contrary that there are two different cycles in $T \cup e$. Then it must be that e belongs to both cycles because there is no cycle in T. We claim that after removing the edge e the remaining graph (that is T) must have a cycle. This is because in the remaining graph there are at least two different paths between the two endpoints of e.

   b) Show that if T is a tree and we add to T k red edges that are not in T, then the resulting graph has at most $2^k - 1$ distinct cycles.

   Hint: show that it is not possible that two different cycles use the same set of red edges.

Solution:It is enough to prove the hint. We prove by induction that if two cycles use the same set of red edges, then there is a (different) cycle containing only non-red edges. This of course will lead to a contradiction because we know there is no such cycle in T. We prove our claim by induction on the number of the non-red edges involved in both cycles. If all the edges in the two cycles are red, then clearly there is only one cycle possible. This is because if there is a cycle made of red edges only, one cannot use the same set of edges to create a different cycle.

Assume there are two different cycles using the same set of red edges. If there is an edge that is not red and is common to both cycles, then color it red and conclude by induction that there is a cycle in the remaining set of non-red edges.

The set of red edges is just a union of disjoint paths. Color blue the non-red edges of the first cycle and color green the non-red edges of the second cycle. Start from an end of one red path and go along blues edges until we get to another end of a red path. Then continue along green edges until we get to another end of a red path, then continue along blue edges until the next end of a red path. If we did not go back to a vertex we already visited before, then we can continue. When we stop we arrived at a vertex we visited before and we closed a cycle made only of blue and green edges.

3. a) It is known that T is a tree with 10 vertices of degree 10 and all other vertices are leaves. How many vertices does T have?

   Solution: Let us denote the number of leaves of T by x. The number of vertices of T is then $10+x$. Now the sum of the degrees of all vertices in T is $100+x$ which should be equal to $2(10+x-1)$. This gives $x=82$.

   b) How many different trees on n labeled vertices are there such that the degree of each vertex is either 3 or 1?

   Solution: Let us denote by x the number of vertices of T of degree 3. T has $n-x$ vertices of degree 1 (the leaves). The sum of degrees of all vertices in T is equal to $3x+(n-x)$. On the other hand this should be equal to $2n-2$. This gives $x=(n-2)/2$. In particular n has to be even, or else there is no such tree.

   We first choose the $(n-2)/2$ vertices out of $v_1, \ldots, v_n$ that will serve as those vertices with degree 3. This can be done in $\binom{n}{(n-2)/2}$ ways.

Then We need to construct a sequence of length $n-2$ containing two copies of each of these vertices that we chose. This can be done in $(n-2)!2^{-(n-2)/2}$ ways. Altogether the answer is: $\binom{n}{(n-2)/2}(n-2)!2^{-(n-2)/2}$.

4. Show that when n is even, then the complete graph K_n (that has $(n-1)n/2$ edges) is a union of $n/2$ trees on the same set of vertices. In other words: show that the set of edges of the complete graph K_n can be partitioned into $n/2$ sets of $n-1$ edges such that each set of $n-1$ edges forms a tree on the set of vertices of K_n.

   Hint: there is more than one way to do it. One way is induction.

   Solution: By induction. We remove two vertices a and b of K_n. The remaining graph K_{n-2} is a union of $n/2-1$ trees on the $n-2$ vertices of K_{n-2}. For each of the $n/2-1$ trees we add two edge connecting a and b as leaves to two different vertices of the tree. We do it in such a way that each one of the $n-2$ vertices of K_{n-2} will be chosen once in some tree.

   Then we construct an additional tree by taking the edge ab together with all the edges that have not been used yet between a and b and the rest of the $n-2$ vertices. It never happens that we connect a vertex v to both a and b because one of va or vb was used already in the first $n/2-1$ trees. The constructed graph is indeed a tree where apart from a and b all the other vertices are leaves.