Graph Theory 2023 (EPFL): Problem set of week 2

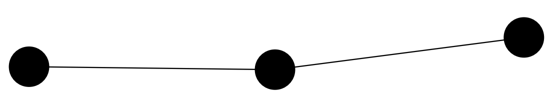
October 4, 2023

1. Let G be the graph that is a cycle of length 4. Find all the eigenvalues and eigenvectors of $A(G)$ and determine the number of walks of length 100 between every two distinct vertices of G.

   Solution: Write down $A(G)$ and find its eigenvalues. These are $0, -1$, $(1+\sqrt{5})/2$, and $(1-\sqrt{5})/2$. Find the corresponding eigenvectors (they are mutually perpendicular).

   We can then write $A(G) = UDU^t$, where D is a diagonal matrix with the eigenvalues on the diagonal. U is a unitary matrix whose columns are the eigenvectors of $A(G)$, normalized. Then $A(G)^1 00$ is equal to $UD^{100}U^t$. You don't get a very clean answer, but still a nice expression involving only powers of the eigenvalues. It is very similar to finding the general element in the Fibonacci sequence and one can indeed show a relation between the two problems.

2. Let G be a path of length 2. That is, G is as in the following figure:

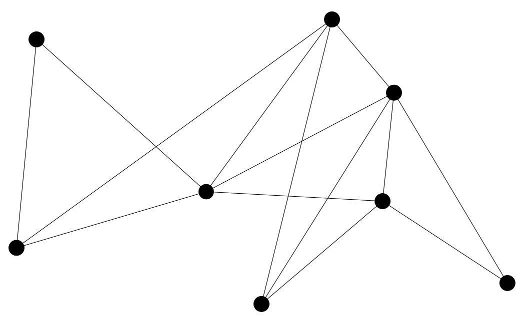
   In how many ways one can get from one end of the path to the other in 100 steps? In 2023 steps?

   Solution: It is very easy to see (why?) that to get from one end to the other we must do an even number of moves. Therefore, there is no way to get from one end to another in 2023 steps.

   To solve the 100 steps case we do the same as in Problem 1. We find the eigenvectors and eigenvalues of $A(G)$ (eigenvalues are $0, \sqrt{2}$, and -$\sqrt{2}$ and follow the same procedure as in Problem 1. It is easier now.

One can also solve Problem 1 and Problem 2 with a completely different approach by recursion formulas. But we are into adjacency matrices of graphs now.

3. Solve the light bulbs puzzle that we saw in class for the following specific graph. We imagine that there is a light bulb in each vertex. Pressing the switch at a vertex changes the condition of the vertex and its neighbors. Find a way to turn ON all light bulbs. At the start they are all OFF.



Solution: Either try all the 256 possibilities, or write down 8 linear equations in 8 variables over $\mathbb{Z}_2$. It should not be difficult to solve them by Gaussian elimination. We are guaranteed that there is a solution.

4. Solve the following puzzle that is a variation of the light bulbs puzzle that we saw in class:

We get a graph G and at every vertex there is a light bulb and a switch. Some of the switches are red and some are blue. If we press a red switch it changes the condition of the light bulbs of the vertex and all its neighbors in G. If we press a blue switch then it changes the condition only of the neighbors of the vertex but not the bulb of the vertex itself. Show that we can always make it happen that precisely all the light bulbs with the red switches are turned ON and the light bulbs with the blue switches are turned OFF. When we start everybody is turned OFF.

Solution: The problem is equivalent to taking $A(G)$ and adding 1's on the main diagonal for the vertices with the red switches. We get a matrix M. Now we want to show that the vector w that has 1's on the "red" coordinates and 0's on the "blue" coordinates is in the image of the matrix M (the space spanned by the columns of M).

We follow the proof we gave in class: The vector w is in the image of M if and only if every vector that is perpendicular to the columns of M is also perpendicular to w.

Assume v is perpendicular to the columns of M. Then $v^t M = 0$. It follows that also $v^t M v = 0$. However, $v^t M v$ is equal to the scalar product of v and w. (Reminder why: $v^t M v = \sum_{i,j} v_i v_j M_{ij}$ and over $\mathbb{Z}_2$ and because M is symmetric this is equal to $\sum_i v_i^2 M_{ii}$. However, $v_i^2 = v_i$ in $\mathbb{Z}_2$. Therefore, it is equal to $\sum_i v_i M_{ii}$. This is the scalar product of v and w because the main diagonal of M is equal to w).