Graph Theory 2023 (EPFL): Problem set of week 6

October 27, 2023

1. Let G be a bipartite graph $V(G) = A \cup B$ such that $|A| = n$ and $|B| = m$. Show that if G does not contain a cycle of length 4, then the number of edges in G is at most $10nm^{1/2} + 10m$.

   Solution: We modify a little bit the proof given in class for general graphs.

   We count the number of triples b, x, y such that $b \in B$, $x, y \in A$ and both x and y are neighbors of b.

   On one hand this is precisely $\sum_{b \in B} \binom{d(b)}{2}$. On the other hand it is at most $\binom{n}{2}$.

   We observe that $\binom{d(b)}{2} \geq (d(b) - 1)^2/2$. Let $d_1, \ldots, d_m$ denote the degrees of the vertices in B. We get $\sum_{i=1}^{m}(d_i - 1)^2/2 \leq n^2$.

   We use $(\sum_{i=1}^{m}(d_i - 1))^2 \leq m \sum_{i=1}^{m}(d_i - 1)^2$

   We get $E = \sum_{i=1}^{m} d_i \leq (m + \sqrt{m}n)$.

2. Let H be a bipartite graph. Prove that there is $\epsilon > 0$ such that $Ex(H, n) \leq c_k n^{2-\epsilon}$. In other words, $Ex(H, n)$ is subquadratic for every bi-partite graph H.

   Solution: Write $V(H) = A \cup B$. Then in particular G cannot contain $K_{|A|,|B|}$ because the complete bipartite graph on $|A|$ and $|B|$ vertices will definitely contain a copy of H. From Kovari-Sos-Turan Theorem it now follows that the number of edges in G is at most $cn^{2-\frac{1}{\min(|A|,|B|)}}$.

3. Prove that for every n nonnegative numbers $a_1, \ldots, a_n$ we have $\frac{1}{n}\sum_{i=1}^{n} a_i \leq \sqrt[k]{\frac{1}{n}\sum_{i=1}^{n} a_i^k}$.

Solution: There is more than one way to show this. It is not the easiest inequality but not the most difficult either.

We want to show that $(\sum a_i)^k \leq n^{k-1} \sum a_i^k$. Keeping the sum $\sum a_i$ fixed, $\sum a_i^k$ is minimized where all the a_i's are equal. It is not difficult to see this. It is enough to prove this when $n = 2$. In other words the function $x^k + (c-x)^k$ is minimized for $x = c/2$. This is easily checked by looking at the derivative and equating it to 0.

When all a_i's are equal we get equality.

4. We have seen in class that if T is a tree with k vertices, then $Ex(T,n) \leq 10k^2 n$. Improve on the dependency in k of this bound and show that $Ex(T,n) \leq 10kn$.

   Hint: use the result we showed in class allowing to assume that the degree of every vertex is at least half of the average degree.

   Solution: Assume G has more than $10kn$ edges. Find a sub-graph G' of G with average degree $20k$ and minimum degree $10k$. We claim that G' contains a copy of every tree with k or less vertices. This is true for any tree with one vertex. Assume it is true for i we prove it for $i+1$. Let H be a tree with $i+1$ vertices. Remove a leaf v from H and obtain a tree H' with i vertices. By induction hypothesis G' contains a copy of H'. The vertex y in this copy of H' that is the neighbor of v in H has degree at least $10k$ in G'. We can therefore find a "free" edge going out from y in G' to a new vertex not in the copy of H' in G'. We therefore found a copy of H in G'.

5. Let G be a graph on n vertices that does not contains a cycle of length 5. We know already that G may have even $n^2/4$ edges. Show that it cannot have more than $n^2/4 + 100n$ edges.

   Solution: Let a be the vertex with largest degree in G and denote the set its neighbors by B. Then for every other vertex v not in B we delete all edges incident to v and connect v to every vertex in B. By doing this we only increase the number of edges in the graph.

   We get a complete bipartite graph (that contains at most $n^2/4$ many edges) plus some edges between vertices in B. Notice that the edges between vertices in B cannot create a path of length 3 because then we had a cycle of length five. This implies that the number of edges between vertices in B is at most $100n$ (and in fact much less, why?). Altogether, G cannot have more than $n^2/4 + 100n$ edges.