

Graph Theory 2023 (EPFL): Problem set of week 6

October 26, 2023

1. Let G be a bi-partite graph $V(G) = A \cup B$ such that $|A| = n$ and $|B| = m$. Show that if G does not contain a cycle of length 4, then the number of edges in G is at most $10nm^{1/2} + 10m$.
2. Let H be a bipartite graph. Prove that there is $\epsilon > 0$ such that $Ex(H, n) \leq c_k n^{2-\epsilon}$. In other words, $Ex(H, n)$ is subquadratic for every bi-partite graph H .
3. Prove that for every n nonnegative numbers a_1, \dots, a_n we have

$$\frac{1}{n} \sum_{i=1}^n a_i \leq \sqrt[k]{\frac{1}{n} \sum_{i=1}^n a_i^k}.$$

4. We have seen in class that if T is a tree with k vertices, then $Ex(T, n) \leq 10k^2n$. Improve on the dependency in k of this bound and show that $Ex(T, n) \leq 10kn$.
Hint: use the result we showed in class allowing to assume that the degree of every vertex is at least half of the average degree.
5. Let G be a graph on n vertices that does not contains a cycle of length 5. We know already that G may have even $n^2/4$ edges. Show that it cannot have more than $n^2/4 + 100n$ edges.