Graph Theory 2023 (EPFL): Problem set of week 5

October 20, 2023

1. Show that if G does not contain a cycle of even length, then no two cycles in G may share an edge.

   Solution: Suppose c_1 and c_2 are two cycles (of odd length) in G that share at least one edge. If c_1 and c_2 have the same set of edges, then it is the same cycle. Therefore, there must be an edge say of c_2 that is not an edge of c_1. Remove from c_2 all the edges that are also in c_1. There is at least one such edge. What remains of c_2 is a union of paths. Take one such path. This path P is between two vertices x and y of c_1. x and y divide c_1 into two paths one of which has even length and one has odd length. Both of this paths create a cycle together with P. One of the cycles will have even length and one will have odd length. This is a contradiction because we assume G does not contain an even cycle.

2. Let G be a graph on n vertices. Consider the matrix E as we defined in class where the rows correspond to the vertices and the columns to the edges. We make a small change and each column has two 1's at the entries that correspond to the vertices of the edge (rather than $+1$ and -1 as we had in class).

   Show that the determinant of n columns of E is non-zero if and only if the corresponding n edges form a subgraph H of G that has precisely one cycle, necessarily of odd length, in each connected component of H.

   Solution: Denote by M the matrix that is composed of the columns of E corresponding to the edges in H. If one of the connected components of H is a tree T say with k vertices, then M must be singular. This is because the rows corresponding to the vertices of T have non-zero entries only in $k-1$ columns. This implies that the rank of the k rows corresponding to the vertices of T is at most $k-1$.

It now follows that every connected component of H must contain a cycle. Because H has n vertices and n edges, then it must be that every connected component of H has the same number of edges as vertices. As we have seen already, this implies that each connected component of H has precisely one cycle. If such a cycle has even length, then we claim that M is singular. This is because the vector v that has 0's in all the entries corresponding to vertices not in the cycle and has $+1$ and -1 alternatingly for vertices on the cycle satisfies $vM = 0$. On the other hand if the cycle has odd length We claim that M is non-singular. There is more than one way to see this. For example: remove from M a column u that correspond to an edge e of the cycle. Then we know from the Matrix-Tree Theorem that the rank of the remaining matrix M' is equal to $n-1$. Let v be the only vector such that $vM' = 0$. The values of every pair of coordinates of v that correspond to neighboring vertices in G must be negative of one another. If we consider the edges of the cycle minus the edge e we see that the two coordinates of v corresponding to the two vertices of e must have the same (non-zero, why?) value, because c has odd length. This means that $vM \neq 0$ and therefore, M is non-singular.

3. Let G be a graph with no cycle of even length. Show that G has at most $\frac{3}{2}n$ edges.

   Solution: Denote by E the number of edges in such a graph G. By the previous problem, no two cycles in G share an edge. Because every cycle has length of at least 3, it follows that the number of cycles in G is at most $E/3$. If we remove one edge of each cycle in G we remain with a tree with $n-1$ edges. Therefore, $E - E/3 \leq n-1$. It follows from here that $E \leq \frac{3}{2}(n-1)$.

4. Let n points be given on a circle of radius 1 in the plane. Show that at least $n^2/10 - \frac{3n}{2}$ pairs of the given points are at distance smaller than 1.

   Solution: Define a graph G where two points (vertices) are connected by an edge if their distance from each other is greater than 1. It is easy to check that G does not contain K_6. Therefore, by Turan's Theorem, the number of edges in G is at most $\frac{n^2}{2}(1 - \frac{1}{6-1}) = \frac{4}{5}\frac{n^2}{2}$. This means that there are at least

$$\binom{n}{2} - \frac{4}{5}\frac{n^2}{2} \geq n^2/10 - 1/2n$$

edges in the complement of G. This means that at least $n^2/10 - 1/2n$ pairs of points are at distance smaller than or equal to 1. We notice that the number of pairs points at distance exactly 1 is at most n. This is because every point is at distance 1 from at most two other points. It follows now that the number of pairs of points at distance smaller than 1 is at least $n^2/10 - 3/2n$.