Graph Theory 2023 (EPFL): Problem set of week 1

September 25, 2023

1. The complement $\overline{G}$ of a graph G is a graph on the same set of vertices such that any two vertices in $V(G) = V(\overline{G})$ are connected by an edge in G if and only if they are not connected by an edge in $\overline{G}$.

   Show that either G, or $\overline{G}$ (or both) must be a *connected* graph. A connected graph is a graph where every two vertices can be connected by a path of edges.

   Solution: If G is not connected, take a vertex x and let A be the set of all vertices that are connected to x by a path in G. A may consist of x alone, but may also be bigger. Let $B = V(G) \setminus A$ and observe that B cannot be empty and let y be a vertex in B.

   We claim that $\overline{G}$ is connected. Take two vertices u and v in $V(G) = V(\overline{G})$. If both u and v are in A, then uyv is a path in $\overline{G}$. If both u and v are in B, then uxv is a path in $\overline{G}$. If one of u and v is in A and the other is in B, then uv is an edge in $\overline{G}$.

2. Two graphs G and H are called isomorphic if there is a one to one and onto map f from $V(G)$ to $V(H)$ such that $x, y \in V(G)$ are connected by an edge in G if and only if $f(x)$ and $f(y)$ are connected by an edge in H.

   For every $n \geq 6$ give an example of two graphs on n vertices that are not isomorphic but have the same set of degrees of vertices.

   Solution: A cycle of length n compared with two smaller cycles of lengths that sum up to n.

3. a) Give two different examples for graphs G that are isomorphic to their complement $\overline{G}$.

Solution: A cycle of length 5. Also a path of length 3 (edges). There are arbitrarily large examples.

b) Can you find a graph G on 2023 vertices that is isomorphic to its complement $\overline{G}$?

Solution: No because G and $\overline{G}$ should have the same number of edges and together they should have $\binom{2023}{2}$ edges, which is an odd number.

4. a) There is a group of n kids. Some pairs of kids are friends, some are not. Show that one can always divide the kids into two groups such that every kid has at least as many friends in the other group than in his/her own group.

   Solution: Start with some partitioning of the kids. If there is a kid that has more neighbors in his group than in the other group, then move the kid to the other group. By doing this we increased the number of pairs of kids who are friends and belong to different groups. Continue in the same way until every kid has at least as many neighbors in the other group than in his/her own group. This must happen at some point because the number of pairs of kids is finite.

   b) Conclude from here (what we have already seen in class) that every graph G contains a bipartite subgraph H with at least half of the edges of G.

   Solution: We think of the vertices of G as the kids and think of the edges as friendship relation. Partition the kids as in the previous problem and observe that the bipartite subgraph of edges that connect kids in different groups contains at least half of the edges of G. This is because the degree of every vertex in the subgraph is at least half of the degree of the vertex in G. This is in fact a stronger result showing that there is a bipartite subgraph H of G where the degree of every vertex in H is at least half of its degree in G.