Integer Optimization
Problem Set 7

Presentations: May 1

i) Consider a line-segment $\ell = \{d\lambda\colon \lambda \in [a,b]\} \subseteq \mathbb{Z}^2$ with $d \in \mathbb{Z}^2 \setminus \{0\}$ and $a, b \in \mathbb{Q}$. Use extended the euclidean algorithm to decide $\ell \cap \mathbb{Z}^2 = \emptyset$ in polynomial time in $\mathrm{size}(d) + \mathrm{size}(a) + \mathrm{size}(b)$.

ii) Consider the problem of deciding $K \cap \mathbb{Z}^n = \emptyset$ for a convex body $K \subseteq \mathbb{R}^n$ and let $a \in \mathbb{Z}^n$ and $\beta \in \mathbb{Z}$. Show that the problem

$$K \cap \{x \in \mathbb{R}^n\colon a^T x = \beta\} \cap \mathbb{Z}^n = \emptyset?$$

can be understood as a problem $K' \cap \mathbb{Z}^{n-1} = \emptyset$ for some convex body $K' \subseteq \mathbb{R}^{n-1}$. If K is a polytope $K = \{x \in \mathbb{R}^n\colon Ax \leqslant b\}$ for some $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$, what is a description of the polytope K'?

iii) Let $a_1, \ldots, a_n$ be a basis of the lattice $\Lambda \subseteq \mathbb{R}^n$, then

$$\mu(\Lambda) \leqslant \frac{1}{2}\sum_{i=1}^{n} \|a_i\|_2.$$

iv) Let $a_1, \ldots, a_n$ be a basis of the lattice $\Lambda \subseteq \mathbb{R}^n$. Let $\Lambda' = \{\pi(v)\colon v \in \Lambda\}$, where $\pi(x) = x - \langle x, a_1\rangle\langle a_1, a_1\rangle a_1$ is the projection of x into the orthogonal complement of a_1.

Show that Λ' is a $n-1$-dimensional lattice (in the hyperplane $a_1^T x = 0$). What is $\det(\Lambda')$?

v) Let $\Lambda(A) \subseteq \mathbb{R}^n$ be a lattice, where $A \in \mathbb{R}^{n\times n}$ is non-singular. Define the set $\Lambda^* = \{y \in \mathbb{R}^n\colon y^T v \in \mathbb{Z}$ for each $v \in \Lambda\}$. Show that Λ^* is a lattice. Determine a basis of Λ^*. What is $\det(\Lambda^*)$?

Λ^* is called the dual lattice of Λ.