Discrete Optimization 2023 (EPFL): Problem set of week 9

May 30, 2023

Reminder: The min-max theorem for zero-sum games with mixed strategies says that for every $m \times n$ matrix A we have

$$\min_x \max_y y^T Ax = \max_y \min_x y^T Ax,$$

where the minimum is over all $y = (y_1, \ldots, y_m) \geq 0$ such that $\sum y_i = 1$. The maximum is over all $x = (x_1, \ldots, x_n) \geq 0$ such that $\sum x_i = 1$.

1. Let A be an $m \times n$ matrix. Assume that there is an entry in A that is the minimum in its row and the maximum in its column. Prove that this entry is the value of the zero-sum game with for two players with mixed strategies.

   Solution: Let the value of that entry be equal to M and assume it is in the ith row and jth columns.

   Consider the case in which the row-player (player R) choose $y = (0, \cdots, 1, \cdots, 0)^T = e_i$, the ith unit vector in $\mathbb{R}^m$. Then no matter what the column player (player C) chooses, the value of the game will be at least M, since

   $$\max_y \min_x y^T Ax \geq \min_x e_i^T Ax = \min_x a_i x = M$$

   where a_i is the ith row of A, and the minimum is reached by taking x to be the jth unit vector in $\mathbb{R}^n$.

   Consider the case in which the column-player(player C) choose $x = (0, \cdots, 1, \cdots, 0)^T = e_j$, the jth unit vector in $\mathbb{R}^n$. Then no matter what player R chooses, the value of the game will be at most M, since

   $$\min_x \max_y y^T Ax \leq \max_y y^T Ae_j = \max_y y^T a'_j = M$$

where a'_j is the jth column of A, and the minimum is reached by taking y to be the ith unit vector in $\mathbb{R}^m$.

We are done because the min-max theorem gives

$$\min_x \max_y y^T Ax = \max_y \min_x y^T Ax,$$

2. Prove the min-max theorem directly for matrices of the form

$$A = \begin{pmatrix} a & b \\ b & a \end{pmatrix}$$

Solution: Consider first $max_x \min_y yAx$. Fixing $x = (x_1, x_2)$, we have $\min_y yAx = \min_y y_1(ax_1 + bx_2) + y_2(ax_2 + bx_1)$. This is just equal to $\min(ax_1 + bx_2, ax_2 + bx_1)$.

We claim that if we wish to maximize this over all $x = (x_1, x_2)$, then the maximum must be when $ax_1 + bx_2 = ax_2 + bx_1$ because if this is not the case we can move weight from x_1 to x_2 (or vice versa) and make the two expressions $ax_1 + bx_2$ and $ax_2 + bx_1$ closer to each other and their minimum increases (because their sum is constant $a + b$). Therefore, $\max_x \min_y yAx$ is attained when $ax_1 + bx_2 = ax_2 + bx_1$ or equivalently $x_1 = x_2 = \frac{1}{2}$. Then $yAx = \frac{a+b}{2}$.

A similar argument shows that $\min_x \max_y yAx$ has the same value.

3. Find the min-max value for the diagonal matrix with $\lambda_1, \dots, \lambda_n$ on the main diagonal.

Solution: If $\lambda_i \geq 0$ and $\lambda_j \leq 0$ (possibly $i = j$), then the min-max value is equal to 0. This is because we can apply Problem 1 on the entry $a_{ij} = 0$ that is the largest in its column and smallest in its row.

Therefore, assume $\lambda_1, \dots, \lambda_n > 0$.

Then $yAx = \sum x_i y_i \lambda_i$. Fixing y we have $\max_x \sum x_i y_i \lambda_i$ is when $x_i = 1$ for the i such that $y_i \lambda_i$ is maximum.

Then $\max_x yAx = \max_i y_i \lambda_i$. Therefore, if we want to find $\min_y \max_x yAx$, we better have y such that $\max_i y_i \lambda_i$ is minimum. Similar to what we did on Problem 2, this happens when all the $y_i \lambda_i$ are equal. Then $y_i = \frac{1}{\lambda_i}$ times a constant that does not depend on i. Because $\sum y_i = 1$, then we must have

$$y_i = \frac{1}{\sum \frac{1}{\lambda_j}} \frac{1}{\lambda_i}.$$

Then the value of the min-max is $\frac{1}{\sum \frac{1}{\lambda_j}}$

What if the λ_i's are all negative? In this case notice that we have (we use the fact that $\min f = -\max(-f)$)

$$
\begin{aligned}
\frac{1}{\sum \frac{1}{\lambda_j}} &= -\min_y \max_x y(-A)x \\
&= -\min_y \max_x -yAx \\
&= -\min_y(-\min_x yAx) \\
&= \max_y \min_x yAx \\
&= \max_y \min_x xAy \\
&= \min_x \max_y xAy \\
&= \min_y \max_x yAx
\end{aligned}
$$

4. Show that in a zero-sum game with a matrix A with mixed strategies the following is true: If one player knows the mixed strategy of the other player, then the best response (strategy) for him is a pure strategy. That is, the best response is choosing just one row or column.

   Solution: Assume for example that the Column player knows the strategy y_0 of the Row player. Then the Column player wants to maximize $\max_x y_0 Ax$ subject to $\sum x_i = 1$ and $x \geq 0$. However, this is a linear program. The simplex algorithm will find a vertex of the polyhedron $\sum x_i = 1$ and $x \geq 0$. The vertices of this polyhedron are the n vectors that have all their coordinates equal to 0 except one coordinate that is equal to 1.