Discrete Optimization 2023 (EPFL): Problem set of week 13

May 30, 2023

1. Consider the polyhedron P defined by $Ax \leq b$ for

$$A = \begin{pmatrix} 1 & 3 & 2 \\ 2 & 0 & 1 \\ -1 & 1 & -2 \\ 3 & 1 & -1 \\ 2 & 2 & 1 \end{pmatrix}$$

   and $b = (1, 2, 3, 4, 5)$.

   Find a hyperplane (weakly) separating P and $x = (1, 2, 3)$.

   Solution: Calculating Ax we find $Ax = (13, 5, -2, 2, 9)$. We see that Ax does not satisfy $Ax \leq b$ because of the first, second, and fifth coordinates.

   We can therefore take any of the first, second, or fifth constraints and each of them will separate x from P. For example we can take the hyperplane $H = \{2x + 2y + z = 5\}$.

2. Find the volume of the cross polytope, the polytope with the $2n$ vertices in $\mathbb{R}^n$ that are $\pm e_1, \ldots, \pm e_n$.

   Solution: The answer is $\frac{2^n}{n!}$. The reason is that the cross-polytope is the disjoint union of 2^n pyramids generated by the origin and $\pm e_1, \ldots, \pm e_n$. Their volume is each $\frac{1}{n!}$.

3. Let P be the tetrahedron with $n+1$ integer vertices in $\mathbb{R}^n$. Assume that P contains an integer interior point. Prove that the volume of P is at least $\frac{n+1}{n!}$.

Solution: We consider each of the $n+1$ simplices generated by the interior point and one of the $(n-1)$-dimensional faces of the tetrahedron. The volume of each one is at least $\frac{1}{n!}$ since the vertices of P are integer. These $n+1$ simplices are disjoint.

4. Let T be an integral non-singular linear transformation. Assume that the entries in the matrix representing T are at most D in absolute value. By how much can T shrink the size of a vector at most? That is, how small can $\frac{|Tx|}{|x|}$ be?

   Solution: If T is a non-singular linear transformation represented by a matrix whose entries are all smaller than D, then T cannot increase the size of a vector by more than a factor that is larger than nD. Let $A = T^{-1}$. Then $\frac{|Tx|}{|x|} \leq \frac{|Tx|_\infty}{|x|_\infty}$ for any norm $|\cdot|$. Now $\frac{|Tx|}{|x|} \leq \frac{|Tx|_\infty}{|x|_\infty} \leq \frac{n\Delta|x|_\infty}{|x|_\infty}$ where Δ is the maximum entry of A. Then it follows that $\frac{|Tx|}{|x|} = \frac{|Tx|}{T^{-1}T|x|} = \frac{y}{T^{-1}y}$ for $y = Tx$. Therefore, $\frac{|Tx|}{|x|} = \frac{y}{T^{-1}y} \geq \frac{1}{\Delta n}$. Now we want to determine Δ. By Hadamard‘s inequality, $T_{ij}^{-1} = \frac{Det_{ij}}{Det(T)} \geq (n-1)!D^{n-1}$ since each of the entries of T is upper bounded by D. Thus, $\frac{|Tx|}{|x|} \geq \frac{1}{n!D^{n-1}}$.