

# Discrete Optimization 2023 (EPFL): Problem set of week 12

May 25, 2023

1. Verify that  $\{(x, y) \in \mathbb{R}^2 \mid 5x^2 - 2xy + 2y^2 \leq 1\}$  is an ellipse and find its area.

**Solution:** We can write  $5x^2 - 2xy + 2y^2 = (x + y)^2 + (2x - y)^2$ . Therefore,  $T(x, y) = (x + y, 2x - y)$  takes the ellipse to the unit disc (of area  $\pi$ ). We see from here that  $T^{-1}$  takes the unit disc to the ellipse. Therefore, we have an ellipse of area  $\pi |\det(T^{-1})| = \pi \frac{1}{3}$ .

2. Let  $B$  be the unit ball in  $\mathbb{R}^3$  and let  $H$  be the hyperplane  $H = \{x + 3y + 2z = 0\}$ .

Consider the linear transformation

$$T = \begin{pmatrix} 1 & 1 & 2 \\ -1 & 0 & 1 \\ 2 & 1 & 0 \end{pmatrix}$$

What is the (two dimensional) area of  $T(B \cap H)$ ?

**Solution:** First we find two linearly independent vectors in  $H$ , say,  $u = (2, 0, -1)$  and  $v = (3, -1, 0)$ . We calculate the area of the parallelogram they generate (which equals to  $\sqrt{|u|^2|v|^2 - \langle u, v \rangle^2}$ ) and we get  $\sqrt{14}$ . Then we consider  $T(u) = (0, -3, 4)$  and  $T(v) = (2, -3, 5)$  and compute the area of the parallelogram they generate:  $\sqrt{109}$ . Therefore, the area of  $T(B \cap H) = \frac{\sqrt{109}}{\sqrt{14}}\pi$ .

3. Let  $E \subset \mathbb{R}^3$  be the ellipsoid  $E = \{(x, y, z) \mid \frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{25} \leq 1\}$ .

Let  $H$  be the hyperplane  $H = \{(x, y, z) \mid x - 3y + 2z = 0\}$ .

Find the area of  $E \cap H$ .

**Solution:** The linear transformation  $T(x, y, z) = (\frac{x}{2}, \frac{y}{3}, \frac{z}{5})$  takes the ellipsoid  $E$  to the unit ball.  $T$  takes  $H$  to the hyperplane  $H' = \{2x - 9y + 10z = 0\}$ . We have  $T^{-1}(B \cap H') = E \cap H$ . Now to get the area of  $E \cap H$  calculate the area of  $T^{-1}(B \cap H')$  as we did in the previous problem.

4. Let  $E \subset \mathbb{R}^3$  be the ellipsoid  $E = \{(x, y, z) \mid x^2 + \frac{y^2}{4} + \frac{z^2}{9} \leq 1\}$ .

Let  $H^+$  be the half-space  $H^+ = \{(x, y, z) \mid x + y + z \geq 0\}$ .

Find an ellipsoid  $E'$  such that  $E' \supset E \cap H^+$  and  $Vol(E') \leq Vol(E)e^{-1/(2(3+1))}$ .

**Solution:** The linear transformation  $T(x, y, z) = (x, \frac{y}{2}, \frac{z}{3})$  takes the ellipsoid  $E$  to the unit ball  $B$ . Notice that  $T$  takes  $H^+$  to the half-space  $G^+ = \{x + 2y + 3z \geq 0\}$ . Now we find a linear transformation  $R$  that will rotate the space about the origin such that  $B$  will remain fixed but  $G^+$  will go to the half space  $\{x \geq 0\}$ . This part is technical. One elegant way to do this is to take the vector  $u = \frac{1}{\sqrt{14}}(1, 2, 3)$  and complete it to an orthonormal basis: take also the vector  $v = \frac{1}{\sqrt{5}}(-2, 1, 0)$  and the vector  $w$  perpendicular to both  $u$  and  $v$  (you need to find it). Then consider the linear transformation  $R$  that takes  $u, v, w$  to  $(1, 0, 0), (0, 1, 0),$  and  $(0, 0, 1)$ . This is the inverse of the matrix whose columns are  $u, v, w$ .

The linear transformation  $R(T(\cdot))$  takes  $E \cap H^+$  to the half ball  $B \cap \{x \geq 0\}$ . By the half-ball theorem that we proved in class, the ellipsoid  $F = \{\frac{16}{9}(x - \frac{1}{4})^2 + \frac{8}{9}y^2 + \frac{8}{9}z^2 \leq 1\}$  contains the half ball  $B \cap \{x \geq 0\}$  and we also have  $Vol(F) \leq e^{\frac{-1}{2(3+1)}} Vol(B)$ .

If we now apply  $T^{-1}(R^{-1}(\cdot))$  on  $F$  we get the desired ellipsoid  $E'$ . One can get from here the algebraic description of  $E'$ . Technichally, we need to replace  $x, y,$  and  $z$  in the algebraic discription of  $F$  by  $R(T(x, y, z))$ .