Discrete Optimization 2023 (EPFL): Problem set of week 11

May 16, 2023

Reminder: Hall's theorem for bi-partite graphs: If we have a bi-partite graph with parts $A \cup B$, then it has a matching for all the vertices in A if and only if for every subset $A' \subset A$ we have that the total number of vertices in B connected to at least one vertex in A' is at least the size of A'. This problem set includes some applications of this theorem.

1. We saw that in bipartite graph the maximum size of a matching is equal to the minimum size of a vertex cover. In general graphs the minimum vertex cover is greater than or equal to the maximum size of a matching. Show that it is always true that the minimum vertex cover is at most twice the size of the maximum matching in a graph. For every n find a graph with maximum matching equal to n and minimum vertex cover equal to $2n$.

2. Let G be a bipartite graph where every vertex has the same degree d (such graphs are called d-regular). Show that the edges of G can be partitioned into d sets, each of which is a matching.

3. Let A be an $m \times n$ matrix such that each of the numbers $1, 2, \dots, n$ appear precisely m times as an entry in A. Show that we can permute within each column separately such that in the resulting matrix every row contains all the numbers $1, 2, \dots, n$.

4. We have 100 boxes, each is locked with a lock. We also have the 100 keys for the locks but we don't know which key opens which lock. In every round we can try each key (possibly more than one key) on one lock only but in such a way that we do not try two different keys on the same lock. Our goal is to open at least one box. Show that there

exists a strategy such that 51 rounds are enough. Furthermore, show that 50 rounds are not always enough.