Discrete Optimization 2023 (EPFL): Problem set of week 11

May 16, 2023

Reminder: Hall's theorem for bi-partite graphs: If we have a bi-partite graph with parts $A \cup B$, then it has a matching for all the vertices in A if and only if for every subset $A' \subset A$ we have that the total number of vertices in B connected to at least one vertex in A' is at least the size of A'. This problem set includes some applications of this theorem.

1. We saw that in a bipartite graph the maximum size of a matching is equal to the minimum size of a vertex cover. In general graphs the minimum vertex cover is greater than or equal to the maximum size of a matching. Show that it is always true that the minimum vertex cover is at most twice the size of the maximum matching in a graph. For every n find a graph with maximum matching equal to n and minimum vertex cover equal to $2n$.

   Solution: Assume the maximum matching has size k. Consider a matching of size k and then the $2k$ vertices of the k edges participating in that matching are a vertex cover. Indeed, if there is an edge both of whose vertices are not among the $2k$ vertices of the maximum matching, then we could add this edge to the maximum matching and get a larger matching, which is a contradiction. A graph that consists of n disjoint triangles is an example for a graph with maximum matching n and minimum vertex cover $2n$.

2. Let G be a bipartite graph where every vertex has the same degree d (such graphs are called d-regular). Show that the edges of G can be partitioned into d sets, each of which is a matching.

   Solution. Denote the two parts of the bipartite graph G by A and B. The number of edges in G is equal to $d|A|$ and also to $d|B|$. Hence

$|A| = |B|$. We first find a matching of size $|A| = |B|$ by using Hall's theorem. Consider any subset A' of A of size k. There are kd edges going out from A'. Let B' denote the set of all vertices in B connected to some vertex in A'. The number of edges going out from B' is $d|B'|$. They include all the edges going out from A'. Therefore, $d|B'| \geq kd$. This implies $|B'| \geq k$. By Hall's theorem there is a matching in G of size $|A|$. If we take out this matching from G we get a $(d-1)$-regular graph and we can finish by induction on d.

3. Let A be an $m \times n$ matrix such that each of the numbers $1, 2, \ldots, n$ appear precisely m times as an entry in A. Show that we can permute within each column separately such that in the resulting matrix every row contains all the numbers $1, 2, \ldots, n$.

   Solution. We define a bipartite graph, one part of size n represents the columns. The other part is of size n and represents the numbers $1, 2, \ldots, n$. We connect a column to a number if the number appears in that column. We claim that we can find a matching of size n. This would mean that we can choose for each column a number appearing in this column such that every column has a different number. We can then put these numbers on the first row of the matrix and then consider the rest of the rows and continue by induction on m.

   To see that there is a matching of size n we use Hall's theorem. Consider any k columns of the matrix. We need to show that in total we see at least k distinct numbers in all these columns. Indeed, it must be the case or else there is a number that appears more than m times (the k columns contain together km numbers and if there are less than k distinct numbers there, then at least one number appears more than m times).

4. We have 100 boxes, each is locked with a lock. We also have the 100 keys for the locks but we don't know which key opens which lock. In every round we can try each key (possibly more than one key) on one lock only but in such a way that we do not try two different keys on the same lock. Our goal is to open at least one box. Show that there exists a strategy such that 51 rounds are enough. Furthermore, show that 50 rounds are not always enough.

   Solution. To see that we can do with 51 rounds, take any 51 keys and then it must be that one of them opens one box from the first 51 boxes (otherwise there are only 49 keys left to open the first 51 boxes which is impossible). Then in 51 rounds we can try all the 51 keys on the 51 first boxes and one box will be opened for sure.

To see that 50 rounds are not enough we think about it in the following way. Draw the complete graph between the keys and the boxes. Every round color by red those edges that correspond to a key that we tried on a box. So every round we color by red a matching of size at most 100. It is enough to show that after 50 rounds we can still find a matching of size 100 of edges that were never colored red.

Notice that after 50 rounds every box is connected by non-red edges to at least 50 keys and every key is connected by non-red edges to at least 50 boxes.

Consider a set A of k boxes and let B be the set of all keys connected to at least one of the boxes in A by a non-red edge. Clearly $|B| \geq 50$ because every box is connected by a non-red edge to at least 50 keys. If $k \leq 50$, then we have $k \leq |B|$, as desired. If $k > 50$, then we claim that $|B| = 100$. Indeed, every key has at least 50 boxes to which it is connected by a non-red edge. One of them must be in A because $|A| \geq 51$. We have shown therefore that the conditions in Hall's theorem are satisfied and we can find a matching of size 100 among the edges not colored red. If this is the "right matching" of keys to boxes, then non of the boxes is opened in the first 50 rounds.