Discrete Optimization 2023 (EPFL): Problem set of week 10

May 11, 2023

1. Let A be a matrix where every row looks like $(0, \ldots, 0, 1, \ldots, 1, 0, \ldots, 0)$, or $(1, \ldots, 1, 0, \ldots, 0)$, or $(0, \ldots, 0, 1, \ldots, 1)$, or $(1, \ldots, 1)$. That is, all the 1's appear in one interval. Show that every $k \times k$ submatrix of A has determinant $0, 1$, or -1 (in other words, A is totally unimodular).

   Solution: By induction on the number of rows in A. If the number of rows is one, then the claim is true since the row consists only of 0 and 1 entries. Now we come to the induction step. If there is no 1 entry on the leftmost column of the new row, then the determinant is equal to 0. Otherwise consider all the rows with 1 on the leftmost entry. Then subtract the shortest of these rows (the one with fewest number of 1's) from the other rows. We get another matrix A' with the same property and there is only one 1 entry on the first (leftmost) column. Now we conclude by induction when computing the determinant of A' that is equal to plus or minus the determinant of one principle minor of A'.

2. Let A be a matrix where each column of A contains only 0's except for one coordinate that is equal to 1 and another coordinate that is equal to -1. Show that every $k \times k$ submatrix of A has determinant $0, 1$, or -1 (in other words, A is totally unimodular).

   Solution: We will probably see this also in class: By induction on k. For $k = 1$ this is clear. Let M be a $k \times k$ submatrix of A. If there is a column in M with no ± 1's the determinant is equal to 0. If there is a column in M with only one ± 1, we can conclude by induction on k. Assume therefore that every column of M contains two 1's. Now we can add the rows of M and see that their sum is equal to 0. This means the the columns of M are linearly dependent. Consequently, the determinant of A is equal to 0.

3. Let ℓ be a line through the origin O (could be any other integer point as well). Let $\epsilon > 0$ be any positive number and consider the cylinder S that consists of all points at distance at most ϵ from ℓ. Show that S must contain infinitely many integer points.

   Solution: Let $v = (\alpha_1, \ldots, \alpha_n)$ be a vector in the direction of ℓ. By considering $v, 2v, \ldots, 1000^n v$, there must be two vectors in this list with the same three decimal digits after the decimal point for every coordinate. This means that we can find a and b distinct such that $av - bv$ is very close to an integer point. In fact the distance of $(a-b)v$ from an integer point is at most $\frac{n}{1000}$. Once this is smaller than ϵ we are done because this integer point is in the cylinder. Its distance from $(a-b)v$ is smaller than ϵ. If 1000 is not large enough, take 10^k that is larger than $\frac{n}{\epsilon}$. We can then get more and more such points by taking ϵ to be smaller and smaller.

   For those who are interested: There is a more general theorem (that is not difficult at all) of Minkowski saying that any convex body centrally symmetric about the origin of volume greater than 2^n must contain an integer point different than the origin.

4. Let K be the cone generated by n linearly independent vectors in $\mathbb{R}^n$. Show that K must contain infinitely many integer points.

   Solution. The idea is that very far from the origin the cone has a large volume. In particular it contains a cube that is parallel to the axes and has edge length 1 (or even 1000). This cube must contain an integer point (or even 1000^n).

   To rigorously implement this idea one can take a ray that is contained in the interior of the cone (for example all positive multiples of $W = \sum_{i=1}^n v_i$. Then very far in the cone the cone contains a cylinder of arbitrary large radius around this ray. The cylinder will contain a large cube.

5. Give an example for a linear program with no maximum (in other words, unbounded linear program) such that the corresponding integer program is not unbounded.

   Solution: This is a bit tricky question. One can consider $\max y$ such that $\frac{1}{3} \leq x \leq \frac{2}{3}$ in the two dimensional plane. Clearly there is no maximum. On the other hand there are no integer feasible points at all.

Another example is just to take a line through the origin that does not contain any integer point except for the origin (for example the line spanned by $(1, \sqrt{2})$) and maximize say the x-coordinate of a point on this line.

It is true, however, that once there is an integer point in an unbounded polyhedron with infinite volume, then there are infinitely many integer points there (follows from problem 1 also from problem 2 with some (not very difficult) observations on unbounded polyhedrons.