Discrete Optimization 2023 (EPFL): Problem set of week 13

May 29, 2023

1. Consider the polyhedron P defined by $Ax \leq b$ for

$$A = \begin{pmatrix} 1 & 3 & 2 \\ 2 & 0 & 1 \\ -1 & 1 & -2 \\ 3 & 1 & -1 \\ 2 & 2 & 1 \end{pmatrix}$$

   and $b = (1, 2, 3, 4, 5)$.

   Find a hyperplane separating P and $x = (1, 2, 3)$.

2. Find the volume of the cross polytope, the polytope with the $2n$ vertices in $\mathbb{R}^n$ that are $\pm e_1, \ldots, \pm e_n$.

3. Let P be a tetrahedron with $n + 1$ integer vertices in $\mathbb{R}^n$. Assume that P contains an integer interior point. Prove that the volume of P is at least $\frac{n+1}{n!}$.

4. Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a non-singular linear transformation. Assume that the entries in the matrix representation of T are all integers, and are at most D in absolute value. By how much can T shrink the size of a vector at most? That is, how small can $\frac{\|Tx\|_\infty}{\|x\|_\infty}$ be?

   Hint: Find an upper bound of absolute value of the entries of T^{-1}. You may want to use Hadamard's inequality.