Discrete Optimization 2023 (EPFL): Problem set of week 5

April 25, 2023

1. Consider the simplex (tetrahedron) P in $\mathbb{R}^3$ whose vertices are $(1,0,0)$, $(-1,1,0)$, $(-1,-1,0)$, and $(0,0,1)$. Find all the vectors $\overrightarrow{c}$ such that the maximum of $\langle \overrightarrow{c}, \overrightarrow{x} \rangle$ on P is at the vertex $(0,0,1)$.

   Solution. One possible solution is direct: write $\overrightarrow{c} = (x,y,z)$. We know that the maximum is always obtained at a vertex. Therefore, we need that $z \geq x$, $z \geq y - x$, and $z \geq -x-y$. From the last two inequalities we get $2z \geq -2x$. Now together with the first inequality this implies $z \geq 0$. If $z = 0$, then necessarily $x = 0$ and $y = 0$. We get the vector $(0,0,0)$. This is a "trivial" solution. If $z > 0$ we may assume that it is equal to 1. This is because for any solution $\overrightarrow{c}$ also a positive multiple of it will work. We now get $y - x \leq 1$ and $-x-y \leq 1$. This gives $x \geq -1$. For every $x \geq -1$ we need $-(1+x) \leq y \leq 1+x$ Therefore, the solution are all the positive multiples of $(x,y,1)$ where $x \geq -1$ and $-(1+x) \leq y \leq 1+x$.

   A more generic solution is to find the three vectors that are orthogonal to the three hyper-planes meeting at $(0,0,1)$ and pointing outside of P. Denoting these vectors by v_1, v_2, and v_3, there are b_1, b_2, b_3 such that every point x in P satisfies $\langle \overrightarrow{x}, v_i \rangle \leq b_i$ for $i = 1.2.3$. Denote by H_i the hyper-plane $\langle \overrightarrow{x}, v_i \rangle = b_i$

   Any vector c can be written in one way as $\overrightarrow{c} = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3$. If $\alpha_1, \alpha_2, \alpha_3$ are nonnegative, then $\overrightarrow{c}$ will work for us because $(0,0,1)$ has the maximal scalar product with each of v_1, v_2, and v_3. If one of $\alpha_1, \alpha_2, \alpha_3$ is negative, say $\alpha_1 < 0$, then $\overrightarrow{c}$ is not a solution because by moving from $(0,0,1)$ in a direction that is "away from" (makes an obtuse angle with) v_1 along the intersection of H_2 and H_3, will increase the scalar product with $\overrightarrow{c}$.

The solution is therefore, any positive linear combination of v_1, v_2, and v_3.

2. Let S be the set of all 2^n vectors with coordinates that are equal either to $+1$ or -1. Let P be the set of all linear combinations of the vectors in S with coefficients greater than or equal to 0 and smaller than or equal to 1. Show that P is convex and find at least 3 distinct vertices of P.

   Solution: To show that P is convex, denote the vectors in S by $v_1, \ldots, v_m$, where $m = 2^n$. Consider two points in P, namely $A = \sum_{i=1}^m \alpha_i v_i$ and $B = \sum_{i=1}^m \beta_i v_i$, where $0 \leq \alpha_i, \beta_i \leq 1$ for every i. Now take $0 \leq \lambda \leq 1$ and notice that

$$\lambda A + (1-\lambda)B = \sum_{i=1}^{m} (\lambda\alpha_i + (1-\lambda)\beta_i)v_i.$$

   Observe that $0 \leq \lambda\alpha_i + (1-\lambda)\beta_i \leq 1$ for every i. Therefore, $\lambda A + (1-\lambda)B$ is also in P and consequently, P is convex.

   There is an easy generic way to find vertices in P. Let $\overrightarrow{c}$ be any vector that is not orthogonal to any vector in S (any random vector will do, but one can take for example the vector $\overrightarrow{c} = (1, 10, 10^2, \ldots, 10^m)$.). Let S_+ be the set of all vectors v_i in S such that $\langle \overrightarrow{c}, v_i \rangle > 0$. We claim that $A = \sum_{v_i \in S_+} v_i$ is a vertex of P. Indeed, let $b = \sum_{v_i \in S_+} \langle \overrightarrow{c}, v_i \rangle > 0$. Then the hyperplane $H = \{\langle \overrightarrow{c}, x \rangle = b\}$ contains only the point A. Moreover, every point of $x \in P$ satisfies $\langle \overrightarrow{c}, x \rangle \leq b$. This shows that A is a vertex. We can also write down what the point A is. Notice that the vectors in S_+ are precisely those vectors that end with 1. It is not hard to check that the sum of these vectors is $(0, 0, \ldots, 0, 2^{n-1})$. For similar reasons (and from symmetry) also the vector $(0, 0, \ldots, 0, 2^{n-1}, 0, \ldots, 0)$ is a vertex of P.

3. Let P be the tetrahedron whose vertices are $(1, 2, 3), (2, 1, -1), (1, 1, 0)$, and $(2, 1, -3)$. Find a hyperplane H that supports P and intersects P at the edge with vertices $(1, 2, 3)$ and $(1, 1, 0)$.

   Proof. We first find vectors orthogonal to the hyperplane H_1, through $(2, 1, -1)$, $(1, 2, 3)$ and $(1, 1, 0)$, and the hyperplane H_2, through $(2, 1, -3)$, $(1, 2, 3)$ and $(1, 1, 0)$.

   One can take $u = (-1, 3, -1)$ orthogonal to H_1 and then take $v = (-3, 3, -1)$ orthogonal to H_2. Notice that $H_1 = \{\langle u, x \rangle = 2\}$ and $H_2 = \{\langle v, x \rangle = 0\}$.

Because $\langle u, (2, 1, -3) \rangle < 2$, then indeed every point in P satisfies $\langle u, x \rangle \leq 2$.

Because $\langle v, (2, 1, -1) \rangle < 0$, then indeed every point in P satisfies $\langle v, x \rangle \leq 0$.

As explained in class, every positive linear combination of u and v will result in a vector $\overrightarrow{c}$ that is orthogonal to a hyperplane H, as we need. In particular we can take $\overrightarrow{c} = u + v = (-4, 6, -2)$. Then $H = \{\langle \overrightarrow{c}, x \rangle = 2\}$ will do.

4. Let $\overrightarrow{a}_1, \ldots, \overrightarrow{a}_{n+1}$ be $n + 1$ vectors in $\mathbb{R}^n$ such that every n of them are linearly independent. Let $\overrightarrow{b} \in \mathbb{R}^{n+1}$ be a vector with positive coordinates. Let A be the matrix whose rows are $a_1, \ldots, a_{n+1}$. Show that if $\sum_{i=1}^{n+1} \overrightarrow{a}_i = 0$, then the polyhedron $P = \{x \mid Ax \leq b\}$ is bounded and not empty.

   Proof. Notice that P is never empty because $O \in P$.

   Because $\sum_{i=1}^{n+1} a_i = 0$, every vector can be written as a linear combination of $a_1, \ldots, a_{n+1}$ with nonnegative coefficients. This is because we can replace any $-a_i$ by $\sum_{j \neq i} a_j$.

   In particular $e_1 = (1, 0, 0, \ldots, 0) = \sum_{i=1}^{n+1} \alpha_i a_i$, where $\alpha_i \geq 0$.

   Therefore, for every $x \in P$ we have $\langle e_1, x \rangle \leq \sum_{i=2}^{n+1} \alpha_i b_i$. This shows that the first coordinate of x is bounded from above.

   We can now do the same trick and replace e_1 by $-e_1$ and conclude that the first coordinate of x is bounded from below.

   This applies to every coordinate of x and therefore P is bounded.