Discrete Optimization 2023 (EPFL): Problem set of week 7

April 18, 2023

1. Let P be the unit cube in $\mathbb{R}^n$. That is $P = \{(x_1, \ldots, x_n) \mid 0 \leq x_i \leq 1 \;\; i = 1, \ldots, n\}$. Show that for every $\overrightarrow{c} \in \mathbb{R}^n$ the simplex algorithm will find the maximum of $\langle \overrightarrow{c}, \overrightarrow{x} \rangle$ over all $\overrightarrow{x} \in P$ in at most n iterations (although it has 2^n vertices).

   Solution: Notice that the vertices of the cube are precisely the 2^n vectors $\{(x_1, \ldots, x_n) \mid x_i \in \{0, 1\}\}$. Two vertices are neighbors (adjacent) if and only if they differ in only one coordinate. This is because every $n-1$ supporting hyperplanes with nonempty intersection fix $n-1$ of the coordinates of the two vertices of the same edge of the cube. When we apply an iteration of the simplex algorithm, we move from one vertex to a neighbor of it. We say that the index of this iteration is i if the two vertices differ at the i'th coordinate. We claim that we never have two iterations with the same index. This is because if we have an iteration with index i then c_i must be different from 0 or otherwise this iteration will not improve the value of $\langle \overrightarrow{c}, \overrightarrow{x} \rangle$. If $c_i > 0$, for example, then in this iteration we necessarily change x_i from 0 to 1. This can happen only once. We will never change x_i from 1 to 0 again because it will not improve $\langle \overrightarrow{c}, \overrightarrow{x} \rangle$. Therefore, it cannot be that we will have another iteration with index i. We argue similarly if $c_i < 0$. It now follows that there are at most n iterations.

2. Consider the following (not very difficult) maximization problem: Find $\max \sum_{i=1}^{n} x_i$ subject to $x_i + x_j \leq 1$ for every $i \neq j$.

   What is the dual minimization problem? Try to formulate it in a natural way for a graph on n vertices since there are only n variables in dimension n.

Solution: The linear program here is $\max\{\langle c,x\rangle \mid Ax \le b\}$, where A is the $\binom{n}{2} \times n$ matrix with all the $m = \binom{n}{2}$ possible rows having two 1's and the rest 0's. $b = (1,1,\dots,1)^T \in \mathbb{R}^{\binom{n}{2}}$ and $c = (1,1,\dots,1)^T \in \mathbb{R}^n$.

The dual problem is $\min\{\langle y,b\rangle \mid y^T A = c^T, y \ge 0\}$.

In terms of graphs, let G be the complete graph on n vertices. It has $m = \binom{n}{2}$ edges. The dual problem is to find the optimal way to give nonnegative weights to the edges of the graph such that on one hand the sum of the weights of the edges going from any fixed vertex is equal to 1 and on the other hand we want the sum of the weights of all edges to be minimum.

Notice that for the dual problem all feasible points give the optimal value. This is not the case for the primal problem.

3. Let $\mathcal{F}$ be a family of m subsets of $\{1,\dots,n\}$. We wish to find $x_1,\dots,x_n$ such that $\sum x_i$ is minimum and $\sum_{i\in S} x_i \ge 1$ for every $S \in \mathcal{F}$. Verify that this problem can be written as a linear program. What is the dual (and therefore equivalent) minimization problem?

   Solution: We can write our problem as:

$$-\max\{\sum_{i=1}^{n}(-x_i) \mid Ax \le b\},$$

   where $b = (-1,-1,\dots,-1)^T \in \mathbb{R}^m$ and A is the matrix that has m rows that represent the sets in $\mathcal{F}$. For every $S \in \mathcal{F}$ we have a row with -1 at those coordinates $i \in S$ and 0 otherwise. Notice that $\sum_{i=1}^{n}(-x_i) = \langle c,x\rangle$, where $c = (-1,-1,\dots,-1)^T \in \mathbb{R}^n$. Therefore, our linear problem is $-\max\{\langle c,x\rangle \mid Ax \le b\}$. For convenience denote the sets in $\mathcal{F}$ by $S_1,\dots,S_m$.

   The dual problems is $-\min\{\langle b,y\rangle \mid y^T A = c^T, \quad y \ge 0\}$. We are looking for $-\min \sum -y_i$ where $y^T A = (-1,\dots,-1)$. This is the same as finding the maximum of $\sum y_i$ subject to $y_i \ge 0$ for every i and $\sum_{j\in S_i} y_i = 1$ for every j.

4. Consider the linear program $\max\{\langle c,x\rangle \mid Ax \le b\}$ and assume that it attains a maximum at a single point x at which precisely n constraints meet. Prove that the dual linear problem has a unique minimum.

   Solution: We know from the simplex algorithm that there are $\lambda_1,\dots,\lambda_n > 0$ such that $\sum \lambda_i a_i = c$ for n rows of A that we assume without loss of generality are $a_1^T, \dots, a_n^T$. Notice that $a_1, \dots, a_n$ are linearly independent because x is a vertex.

Consider now the vector $y = (\lambda_1, \dots, \lambda_n, 0, \dots, 0)$. Then $y^T A = c^T$ and hence $\langle b, y \rangle$ is a minimum of the dual problem.

To show that y is the unique minimum, suppose not, then there is another minimum y', which should be positive in at least one coordinates that are equal to 0 in y, since otherwise there is another linear combination of $a_1, \dots, a_n$ that is equal to c. This is impossible because $a_1, \dots, a_n$ are linearly independent.

Now $\langle c, x \rangle = y'^T Ax \leq y'^T b$. On the other hand we assume that $y'^T b$ is also the minimum value of the dual program. Therefore, it must be that x satisfies equality in $Ax \leq b$ for every coordinate at which y' is positive. In particular x has to satisfy equality in another row that is not one of $a_1, \dots, a_n$. This is a contradiction to our assumption that precisely n constraints meet at x.