Linear Programming 2023 (EPFL): Problem set of week 7

April 12, 2023

1. Let P be the unit cube in $\mathbb{R}^n$. That is $P = \{(x_1, \ldots, x_n) \mid 0 \leq x_i \leq 1, \quad i = 1, \ldots, n\}$. Show that for every $\overrightarrow{c} \in \mathbb{R}^n$ the simplex algorithm will find the maximum of $\langle \overrightarrow{c}, \overrightarrow{x} \rangle$ over all $\overrightarrow{x} \in P$ in at most n iterations (although it has 2^n vertices).

2. Consider the following (not very difficult) maximization problem: Find $\max \sum_{i=1}^{n} x_i$ subject to $x_i + x_j \leq 1$ for every $i \neq j$.

   What is the dual minimization problem? Try to formulate it in a natural way for a graph on n vertices.

3. Let $\mathcal{F}$ be a family of m subsets of $\{1, \ldots, n\}$. We wish to find $x_1, \ldots, x_n$ such that $\sum x_i$ is minimum and $\sum_{i \in S} x_i \geq 1$ for every $S \in \mathcal{F}$. Verify that this problem can be written as a linear program. What is the dual (and therefore equivalent) minimization problem?

4. Consider the linear program $\max\{\langle x, \overrightarrow{c} \rangle \mid Ax \leq b\}$ and assume that it attains a maximum at a single point x at which precisely n constraints meet. Prove that the dual linear problem has a unique minimum.