Discrete Optimization 2023 (EPFL): Problem set of week 6

April 4, 2023

1. Let A be an $m \times n$ matrix with rows $a_1, \ldots, a_m$ and let $b \in \mathbb{R}^m$ be given. Consider the polyhedron P defined by $A\overrightarrow{x} \leq \overrightarrow{b}$.

   Assume that $I = \{1, 2, \ldots, n\}$ is a basis, but not a feasible basis. Denote by Q the point that is the intersection of the n hyperplanes $\{\langle a_i, x\rangle = b_i\}$ for $i = 1, \ldots, n$.

   Prove that for every $\lambda_1, \ldots, \lambda_n > 0$ there is α such that the hyperplane $H = \{\langle \sum_{i=1}^n \lambda_i a_i, x\rangle = \alpha$ separates Q and P.

   Solution: Because $a_1, \ldots, a_n$ are linearly independent Q, is the only point that satisfies $\{\langle a_i, x\rangle = b_i\}$ for $i = 1, \ldots, n$. Therefore, for every point of $x \in P$ we have $\langle \sum_{i=1}^n \lambda_i a_i, x\rangle < \sum_{i=1}^n b_i$.

   Let $y \in P$ be the point of maximum of $\langle \sum_{i=1}^n \lambda_i a_i, x\rangle$ over all x in P. We have $\langle \sum_{i=1}^n \lambda_i a_i, y\rangle < \sum_{i=1}^n b_i$. Taking α to be any number between $\sum_{i=1}^n b_i$ and $< \langle \sum_{i=1}^n \lambda_i a_i, y\rangle$ will yield a separating hyperplane.

2. Let P be a (bounded) polytope in $\mathbb{R}^3$ with vertices $v_1, \ldots, v_k$. Let $\overrightarrow{c} \in \mathbb{R}^3$ be such that $\langle c, v_i\rangle \neq \langle c, v_j\rangle$ for every $i \neq j$. Assume that P is a *simple* polytope, in the sense that every vertex has precisely 3 neighbors.

   We say that a vertex v of P is of type 1 if precisely two of its three neighbors have their scalar product with c larger than the scalar product of v and c. We say that v is of type 2 if precisely two of its three neighbors have their scalar product with c smaller than the scalar product of v with c.

   Show that the number of vertices of type 1 is always equal to the number of vertices of type 2. Conclude that every simple polytope in $\mathbb{R}^3$ must have an even number of vertices.

Is it true also in dimensions $2, 4, 5, 6, 7$?

hint: observe that a vertex of type 1 is the "lowest" vertex of precisely one face of dimension 2 of P. A vertex of type 2 is the "highest" vertex of precisely one face of P of dimension 2.

Solution: Every two dimensional face of P is by itself a polytope and it has one vertex with the largest scalar product with c and one vertex with the smallest scalar product with c. Any vertex is a meeting point of three 2-dimensional faces of P (because P is simple). If it is of type 1, it will be the minimum vertex of exactly one of the three faces. If it is of type 2, it will be the maximum in exactly one of the three faces. There is also the maximum vertex of P that is maximum in all three faces meeting there. There is also the minimum vertex of P that is also the minimum of the three faces containing it. Altogether because the number of times a vertex is minimum in a 2-dimensional face is equal to the number of times a vertex is maximum in a 2-dimensional face of P, then the number of vertices of type 1 must be equal to the number of vertices of type 2. This implies that the number of vertices in P is even because there are only two additional vertices the global maximum and the global minimum with respect to the scalar product with c.

By the way, the fact that the number of vertices is even is not surprising because in the graph of neighboring vertices every vertex has degree 3. We know that $3V = 2E$ so V is even. This is true in any odd dimension. But also the result about type 1 and 2 can be generalized to higher dimensions (with more types). This can be done by induction of the dimension. Try it for dimension 4 using the result for dimension 3. You need to consider the 3-dimensional faces of P and for each one we know already what is going on inside it.

Recall that K is a convex set if for every $x, y \in K$ and every $0 \leq \lambda \leq 1$ we have $\lambda x + (1 - \lambda)y \in K$.

3. Let P be a set of vectors (points) in $\mathbb{R}^n$. The *cone* generated by P is the set of all finite sums $\sum a_i p_i$ such that $a_i \geq 0$ for every i and $p_i \in P$. Show that the cone generated by a set of points is always a convex set.

   Solution: Suppose $A = \sum a_i p_i$ and $B = \sum b_i p_i$ are two points in P. Then for every $0 \leq \lambda \leq 1$ we have $\lambda A + (1 - \lambda)B = \sum(\lambda a_i + (1 - \lambda)b_i)p_i$ is by definition also a point in the cone generated by P.

4. Let K be a convex set and assume $p_1, \ldots, p_t \in K$. Show that $\sum \lambda_i p_i \in K$ for every positive $\lambda_1, \ldots, \lambda_t$ whose sum is equal to 1,

   Hint: use induction on t. For $t = 2$ it is just the definition of being convex.

   Solution: By induction on t. $\sum_{i=1}^{t} \lambda_i p_i = (1-\lambda_t)(\sum_{i=1}^{t-1} \frac{\lambda_i}{1-\lambda_t} p_i) + \lambda_t p_t$.

   Notice that $\sum_{i=1}^{t-1} \frac{\lambda_i}{1-\lambda_t} = 1$ by induction hypothesis.

   We may assume $\lambda_t < 1$ for otherwise $\lambda_1 = \ldots = \lambda_{t-1} = 0$ and there is nothing to prove.

5. Given a set P of points, the *convex hull* of P is the set of all finite sums of the form $\sum \lambda_i p_i$ where $\sum \lambda_i = 1$ and the λ_i's are all nonnegative and the p_i's are points in P. Show that the convex hull of a set P of points is a convex set. Conclude it is the smallest convex set containing P.

   Solution: Assume $A = \sum \alpha_i p_i$ and $B = \sum \beta_i p_i$ are two such sums (we can assume the points p_i in both are the same by taking the union). For any $0 \leq \lambda \leq 1$ we have $\lambda A + (1-\lambda)B = \sum(\lambda\alpha_i + (1-\lambda)\beta_i)p_i$. Notice that $\sum \lambda\alpha_i + (1-\lambda)\beta_i = \lambda \sum \alpha_i + (1-\lambda) \sum \beta_i = \lambda + (1-\lambda) = 1$.