Discrete Optimization 2023 (EPFL): Problem set of week 8

April 27, 2023

1. Let K be a cone in $\mathbb{R}^n$. Prove that any hyper-plane H supporting K must pass through the origin O.

   Solution: Let $H = \{\langle q, x\rangle = r\}$ be a supporting hyper-plane for the cone K. We need to show $r = 0$. Let $v \in H \cap K$. We may assume that $\langle q, x\rangle \geq r$ for every $x \in K$, with $r \geq 0$ (otherwise replace q by $-q$ and replace r by $-r$). Because $v \in K$ and K is a cone, then also $\frac{1}{2}v, 2v \in K$. We have $\langle q, \frac{1}{2}v\rangle = \frac{1}{2}r$ and $\langle q, 2v\rangle = 2r$. Hence it must be that $\frac{1}{2}r, 2r \geq r$. This implies $r = 0$.

2. Prove that $A\overrightarrow{x} = \overrightarrow{b}$ has a solution (we do not require $x \geq 0$ as in Farkas' Lemma) if and only if for every y such that $y^T A = 0$ we also have $\langle y, b\rangle = 0$.

   Solution: $Ax = b$ has a solution if and only if $Ax - Az = b$ has a solution with $x, z \geq 0$. By Farkas' lemma applied for the system $A'(x, z) = b$ for $A' = (A, -A)$, this system has a solution if and only if for every y such that $y^T A' \geq 0$ we have $\langle y, b\rangle \geq 0$. Now $y^T A' \geq 0$ is equivalent to $y^T A \geq 0$ and $y^T(-A) \geq 0$. Hence $y^T A = 0$.

   Therefore, the original system has a solution if and only if $y^T A = 0$ implies $\langle y, b\rangle \geq 0$.

   But this is if and only if $y^T A = 0$ implies $\langle y, b\rangle = 0$

   (because if $y^T A = 0$, then also $-y^T A = 0$ and so this should imply $\langle y, b\rangle \geq 0$ and $\langle -y, b\rangle \geq 0$. This is the same as $\langle y, b\rangle = 0$.

3. Prove the following Farkas-like Lemma: $Ax < 0, \;\; x \geq 0$ has a solution if and only if there is no $y > 0$ such that $y^T A \geq 0$.

Solution: $Ax < 0, x \geq 0$ has a solution if and only if $Ax + z = (-\epsilon, \ldots, -\epsilon), x \geq 0 \in \mathbb{R}^n, z \geq 0 \in \mathbb{R}^m$ has a solution for some $\epsilon > 0$.

We apply Farkas' lemma. The matrix of the new system is

$$A' = (A \,|\, I_m).$$

The new system has a solution if and only if $y^T A' \geq 0$ implies also $\langle y, (-\epsilon, \ldots, -\epsilon) \rangle \geq 0$.

Now, $y^T A' \geq 0$ is equivalent to $y^T A \geq 0$ and $y \geq 0$. In such a case in order for y to satisfy $\langle y, (-\epsilon, \ldots, -\epsilon) \rangle \geq 0$ (for some $\epsilon > 0$) we must have $y = 0$. Therefore, having $y^T A' \geq 0$ implying $\langle y, (-\epsilon, \ldots, -\epsilon) \rangle \geq 0$ is the same as $y^T A' \geq 0$ implying $y = 0$. This is the same as $y^T A \geq 0$ and $y \geq 0$ implies $y = 0$.

This is the same as if there is no $y \geq 0, \quad y \neq 0$ such that $y^T A \geq 0$.

This is the same as if there is no $y > 0$ such that $y^T A \geq 0$.

4. Prove the following Farkas-like Lemma: $Ax = 0, \quad x > 0$ has a solution if and only if there is no y such that $y^T A \geq 0$ and $y^T A \neq 0$.

   Solution: $Ax = 0, \quad x > 0$ has a solution if and only if

   $Ax = 0, -x + z = (-\epsilon, \ldots, -\epsilon), x, z \geq 0$ has a solution for some $\epsilon > 0$.

   This is the same as $A'(x, z) = (0, 0 \ldots, 0, -\epsilon, \ldots, -\epsilon)$ has a solution for the matrix

   $A' = \begin{pmatrix} A & 0_{m,n} \\ -I_n & I_n \end{pmatrix} \in \mathbb{R}^{(m+n) \times 2n}$

   By apply Farkas' lemma, this is equivalent to:

   $(q, p)^T A' \geq 0$ implies $\langle (q, p), (0, \ldots, 0, -\epsilon, \ldots, -\epsilon) \rangle \geq 0$

   This is the same as $q^T A - p^T \geq 0, \quad p \geq 0$ implies $\langle p, (-\epsilon, \ldots, -\epsilon) \rangle \geq 0$.

   This is the same as $q^T A - p^T \geq 0, \quad p \geq 0$ implies $p = 0$ (because it is not possible that $p \geq 0$ and $\langle p, (-\epsilon, \ldots, -\epsilon) \rangle \geq 0$, unless $p = 0$).

   This is the same as $q^T A \geq 0$ implies $q^T A = 0$ (because if not, then we could take, for example, $p^T = \frac{1}{2} q^T A$ and then $q^T A - p^T \geq 0$ but $p = \frac{1}{2} A^T q \neq 0$). This is exactly what we want.