Integer Optimization
Problem Set 5

Working session: March 27, Presentations: April 3

In this problem set, we will understand the basic principle of the state-of-the-art method to solve integer programming problems

$$\max\{c^T x\colon Ax = b, 0 \leqslant x \leqslant u, x \in \mathbb{Z}^n\}, \tag{1}$$

where $A \in \mathbb{Z}^{m\times n}$, $b \in \mathbb{Z}^m$, $u \in \mathbb{N}^n$ and $c \in \mathbb{Z}^n$. The running time of the method will be

$$(m\Delta + 1)^{O(m^2)} \operatorname{poly}(\operatorname{size}(b) + \operatorname{size}(u)), \tag{2}$$

where $a_{ij} \leqslant \Delta$ for each i, j. Whether this quadratic dependence on m in the exponent can be improved or not, is a prominent open problem.

i) Let $x \in \mathbb{Z}^n \setminus \{0\}$ satisfy $Ax = 0$ and $x \geqslant 0$. Use the Steinitz lemma to conclude that there exists a $y \in \mathbb{Z}^n \setminus \{0\}$ with $Ay = 0$ and

   1) $0 \leqslant y \leqslant x$ (component-wise), and
   2) $\|y\|_1 \leqslant (m\Delta + 1)^{O(m)}$.

ii) For two vectors $x, y \in \mathbb{Z}^n$ we define the partial order $y \sqsubseteq x$ if for each i, $y_i \cdot x_i \geqslant 0$ and $|y_i| \leqslant |x_i|$ hold. Let $x \in \mathbb{Z}^n \setminus \{0\}$ satisfy $Ax = 0$. Conclude that there exists a $y \in \mathbb{Z}^n \setminus \{0\}$ with

   1) $Ay = 0$,
   2) $y \sqsubseteq x$, and
   3) $\|y\|_1 \leqslant (m\Delta + 1)^{O(m)}$.

iii) Let $\overline{x}, x^* \in \mathbb{Z}^n$ be an optimal and feasible solution of (1) respectively. Show that the difference $\overline{x} - x^*$ can be written as a sum

$$\overline{x} - x^* = \sum_{i\in I} y_i, \text{ where } I \text{ is a finite index set and}$$

   each y_i satisfies

   1) $Ay_i = 0$,
   2) $y_i \sqsubseteq \overline{x} - x^*$, and
   3) $\|y_i\|_1 \leqslant (m\Delta + 1)^{O(m)}$.

iv) Show that a feasible solution x^* of (1) is an optimal solution, if and only if the optimal value of the following integer program is 0.

$$\begin{aligned} \max \quad & c^T x \\ \text{s.t.} \quad & \\ & Ax = 0 \\ & 0 \leqslant x + x^* \leqslant u \\ & \|x\|_1 \leqslant (m\Delta + 1)^{O(m)} \end{aligned}$$

v) Design a dynamic program that solves the integer optimization problem in iv) in the running time (2).

vi) Again, let $\overline{x}, x^* \in \mathbb{Z}^n$ be an optimal and feasible solution of (1) respectively. Referring to iii) show that there exists a subset $\widetilde{I} \subseteq I$ of size at most n and $\mu_i \geqslant 0$, $i \in \widetilde{I}$ with

$$\overline{x} - x^* = \sum_{i \in \widetilde{I}} \mu_i y_i.$$

Hint: Carathéodory's Theorem

vii) Let $D = c^T(\overline{x} - x^*)$. Show that there exists a $i \in \widetilde{I}$ and a $k \in \mathbb{N}$ such that $x^* + 2^k y_i$ is feasible and

$$c^T(\overline{x} - (x^* + 2^k y_i)) \leqslant D(1 - 1/(4n))$$

holds.

viii) Show how to find such a y_i in the running time (1).

ix) Show how to solve the integer program (1) in time (1).