Integer Optimization
Problem Set 4

To be discussed on March 20 and after

Consider an integer program in so-called *equation standard form*

$$\max\{c^T x : Ax = b,\ x \geqslant 0,\ x \in \mathbb{Z}^n\}, \tag{1}$$

with $A \in \mathbb{Z}^{m\times n}$ and $b \in \mathbb{Z}^m$ and let Δ be an upper bound on the absolute value of components of A and b. In this exercise, we will show that, if the IP is feasible and bounded, then there exists an optimal solution x^* with $\|x^*\|_\infty \leqslant (m\Delta)^{O(m)}$. To this end, recall the *Hadamard bound* $|\det(B)| \leqslant \prod_i \|b_i\|_2$ and the *matrix inversion formula* $B^{-1} = \widetilde{B}/\det(B)$, where $B \in \mathbb{R}^{m\times m}$ is invertible, $\widetilde{B}$ is the *cofactor matrix* (or its transpose), and the b_i are the columns of B.

1. Let $B \in \mathbb{Z}^{m\times m}$ be a full-rank matrix with $\|b_{ij}\|_\infty \leqslant \Delta$ for each i, j. Show that $B^{-1} = C/D$, where $0 < D \leqslant (m\cdot\Delta)^{m/2}$ is an integer and $C \in \mathbb{Z}^{m\times m}$ is an integer matrix with $|c_{ij}| \leqslant (m\cdot\Delta)^{m/2}$ for each i, j.
   Let $J \subseteq \{1,\dots,n\}$ be a nonempty index set such that the columns a_j, $j \in J$ are linearly dependent. Show that there exists $g \in \ker(A) \cap \mathbb{Z}^n \setminus \{0\}$ such that $\|g\|_\infty \leqslant (m\Delta)^{2m}$ and $g_k = 0$ for each $k \notin J$.[1]

2. Let $x^* \in \mathbb{Z}^n_{\geqslant 0}$ be an optimal solution of (1). Let $J \subseteq \{1,\dots,n\}$ be the set of indices with $x^*_j > (m\Delta)^{2m}$. Show that the corresponding columns a_j, $j \in J$ are linearly independent.

   Hint: This is a bit like in the proof of the Carathéodory Theorem. It can be considered to be an integer version of it.

   How large can the number of columns n be, if there are no redundant columns?

3. Show that there exists an absolute constant K and an optimal solution x^* that satisfies $\|x^*\|_\infty \leqslant (m\Delta)^{Km}$.

[1] A generous upper bound.

The next exercise shows how to reduce a problem

$$Ax = b, x \in \{0,1\}^n \tag{2}$$

to the solution of one equation

$$a^T x = \beta, x \in \{0,1\}^n, \tag{3}$$

where $A \in \mathbb{Z}^{m\times n}$ $b \in \mathbb{Z}^m$, $a \in \mathbb{Z}^n$ and $\beta \in \mathbb{Z}$. To simplify the reasoning, we assume that each entry of A is ± 1 or 0. Let $\lambda \in \{0,\dots,M\}^m$, $a^T = \lambda^T A$ and $\beta = \lambda^T b$. Furthermore, let $S_1, S_2 \subseteq \{0,1\}^n$ be the set of solutions of (2) and (3) respectively.

4. (a) Show that $S_1 \subseteq S_2$ holds.
   (b) An element $x^* \in \{0,1\}^n$ belongs to $S_2 \setminus S_1$ if and only if $\lambda \perp (Ax^* - b)$.
   (c) Suppose that λ is chosen i.i.d. at random from $\{0,\dots,M\}$ and that $x^* \in \{0,1\}^n \setminus S_1$. Show that
   $$P\left[\lambda \perp (Ax^* - b)\right] \leqslant 1/M.$$

5. (a) Show that the number of vectors $(Ax^* - b)$, $x^* \in \{0,1\}^n$ is bounded by $(2n+1)^m$.
   (b) Show
   $$P\left[S_1 \neq S_2\right] \leqslant (2n+1)^m / M.$$
   (c) For $M = (2n+1)^{2m}$,
   $$P\left[S_1 \neq S_2\right] \leqslant \frac{1}{2n}^m.$$
   How many bits has the number M then, if it is represented in a computer?

This is a simple randomized reduction. There are also simple deterministic reductions. Such a reduction shows that the knapsack problem is NP hard. For the following exercise you can assume the existence of such a deterministic reduction with $S_1 = S_2$ is guaranteed.

6. Show that the *knapsack problem* is NP hard, by providing a reduction from SAT.