Integer Optimization
Problem Set 3

To be discussed on March 13

1. Let $P, Q \subseteq \mathbb{R}^n$. Show that
$$\operatorname{conv}(P) \oplus \operatorname{conv}(Q) = \operatorname{conv}(P \oplus Q).$$
Let P and C be subsets of $\mathbb{R}^n$. Show that one has
$$P_I \oplus C_I \subseteq (P \oplus C)_I.$$
Does equality hold?

The following exercises are deriving the Minkowski-Weyl theorem from *linear programming* theory. Recall the following fact from linear programming. If $\max\{c^T x : x \in \mathbb{R}^n,\ Ax \leqslant b\}$ is feasible and bounded and if $\operatorname{rank}(A) = n$, then there exists an optimal *vertex* solution x^*. I.e., x^* satisfies n linear inequalities - whose left-hand-side vectors are a basis of $\mathbb{R}^n$ - with equality.

2. Let $A \in \mathbb{R}^{n\times m}$, consider the finitely generated cone $\mathscr{C} = \{A\lambda : \lambda \in \mathbb{R}^m_{\geqslant 0}\}$ and let $b \notin \mathscr{C}$. With the separation theorem, there exists a hyperplane $\{y : x^T y = \beta\}$ such that $x^T b > \beta$ and $x^T z < \beta$ for each $z \in \mathscr{C}$.
Show that $\beta \geqslant 0$ (on which side is 0?) and $x^T A \leqslant 0$ hold.
3. Next consider the linear program
$$\max\{b^T x : x \in \mathbb{R}^n,\ A^T x \leqslant 0,\ -1 \leqslant x \leqslant 1\}.$$
Show that this linear program is feasible and bounded ($b \notin \mathscr{C}$!) with optimal solution value > 0 and interpret the optimal solution as a hyperplane, separating b from $\mathscr{C}$. Conclude that there is an optimal vertex solution. How many vertex solutions are there at most?
4. Conclude that $\mathscr{C}$ is polyhedral, i.e. there exists a matrix $C \in \mathbb{R}^{m\times n}$ with $\mathscr{C} = \{x : Cx \leqslant 0\}$. How does the matrix $C \in \mathbb{R}^{m\times n}$ look like. How many rows does it have at most.
5. Let $\max\{c^T x : Ax \leqslant b,\ x \in \mathbb{Z}^n\}$ be an integer program with $A \in \mathbb{Z}^{m\times n}$ and $b \in \mathbb{Z}^m$. Show that this integer program is unbounded if and only if the linear programming relaxation (the condition $x \in \mathbb{Z}^n$ is replaced by $x \in \mathbb{R}^n$) is unbounded and the integer program is feasible.
6. Solve the following knapsack problem by drawing the corresponding directed acyclic graph:
$$\max 12x_1 + 10x_2 + 13x_3 + 8x_4$$
s.t.
$$\begin{aligned} 3x_1 + 5x_2 + 2x_3 + 4x_4 &\leqslant 10 \\ x_1, x_2, x_3, x_4 &\in \{0,1\}. \end{aligned}$$