Integer Optimization
Problem Set 3

To be discussed on March 13

1. Let $P, Q \subseteq \mathbb{R}^n$. Show that
   \[ \text{conv}(P) \oplus \text{conv}(Q) = \text{conv}(P \oplus Q). \]
   Let $P$ and $C$ be subsets of $\mathbb{R}^n$. Show that one has
   \[ P_I \oplus C_I \subseteq (P \oplus C)_I. \]
   Does equality hold?

   The following exercises are deriving the Minkowski-Weyl theorem from linear programming theory. Recall the following fact from linear programming. If $\max\{c^T x : x \in \mathbb{R}^n, Ax \leq b\}$ is feasible and bounded and if $\text{rank}(A) = n$, then there exists an optimal vertex solution $x^*$. I.e., $x^*$ satisfies $n$ linear inequalities - whose left-hand-side vectors are a basis of $\mathbb{R}^n$ - with equality.

2. Let $A \in \mathbb{R}^{n \times m}$, consider the finitely generated cone $\mathcal{C} = \{A\lambda : \lambda \in \mathbb{R}^m_{\geq 0}\}$ and let $b \not\in \mathcal{C}$. With the separation theorem, there exists a hyperplane $\{y : x^T y = \beta\}$ such that $x^T b > \beta$ and $x^T z < \beta$ for each $z \in \mathcal{C}$.
   Show that $\beta \geq 0$ (on which side is 0?) and $x^T A \leq 0$ hold.

3. Next consider the linear program
   \[ \max\{b^T x : x \in \mathbb{R}^n, A^T x \leq 0, -1 \leq x \leq 1\}. \]
   Show that this linear program is feasible and bounded ($b \not\in \mathcal{C}$) with optimal solution value $> 0$ and interpret the optimal solution as a hyperplane, separating $b$ from $\mathcal{C}$. Conclude that there is an optimal vertex solution. How many vertex solutions are there at most?

4. Conclude that $\mathcal{C}$ is polyhedral, i.e. there exists a matrix $C \in \mathbb{R}^{m \times n}$ with $\mathcal{C} = \{x : Cx \leq 0\}$. How does the matrix $C \in \mathbb{R}^{m \times n}$ look like. How many rows does it have at most.

5. Let $\max\{c^T x : Ax \leq b, x \in \mathbb{Z}^n\}$ be an integer program with $A \in \mathbb{Z}^{m \times n}$ and $b \in \mathbb{Z}^m$. Show that this integer program is unbounded if and only if the linear programming relaxation (the condition $x \in \mathbb{Z}^n$ is replaced by $x \in \mathbb{R}^n$) is unbounded and the integer program is feasible.

6. Solve the following knapsack problem by drawing the corresponding directed acyclic graph:
   \[ \max 12x_1 + 10x_2 + 13x_3 + 8x_4 \]
   s.t.
   \[ 3x_1 + 5x_2 + 2x_3 + 4x_4 \leq 10 \]
   \[ x_1, x_2, x_3, x_4 \in \{0, 1\}. \]