Integer Programming

Dynamic Programming

$$\begin{array}{lll} \max & c^T x & \\ A\,x & = & b \\ x & \geq & 0 \\ x & \in & \mathbb{Z}^n \end{array}$$

$m :=$ #Rows of A

$A \in \mathbb{Z}^{\mathbf{m} \times \mathbf{n}}$
$b \in \mathbb{Z}^{\mathbf{m}}$
$||A||_\infty \leq \Delta$
$||b||_\infty \leq \Delta$

$$\begin{array}{lll} \max & c^T x & \\ A\,x & \leq & b \\ x & \in & \mathbb{Z}^n \end{array}$$

Standard Form

$(m \cdot \Delta)^{O(m^2)}$ $\leq (m \cdot \Delta)^{100 \cdot m^2}$

(Papadimitriou 1981)

Inequality Form

$n^{O(n)}\,\mathrm{poly}(\log \Delta, m)$

(Kannan 1986)

This talk:

$(m \cdot \Delta)^{O(m)}$ E. & Weismantel 2018

$(m \cdot \Delta)^{O(m)} \cdot ||b||_\infty^2$ (b unrestricted)

$(\log n)^{O(n)}$ in the case $0 \leq x \leq n$

Open: $2^{O(n)}$

$$A \in \mathbb{Z}^{m \times n}, \quad \|A\|_\infty \le \Delta$$

How many cols? $n \le$

$1 \to \circ$ $\{-\Delta, -\Delta+1, \dots, \Delta\}$ $2\cdot\Delta+1$ choices.

$\circ$

$\circ$

$m \to \circ$

$$n \le (2\cdot\Delta+1)^m \Rightarrow n = \Delta^{O(m)}$$

$(m \cdot \Delta)^{O(m^2)}$ — Papadimitriou 1981

$$\begin{array}{ccc} \max & c^T x & \\ A\,x & = & b \\ x & \geq & 0 \\ x & \in & \mathbb{Z}^n \end{array}$$

Fact: *There exists optimal solution x^* with*

$$\|x^*\|_\infty \leq U = (m \cdot \Delta)^m$$

$\|b\|_\infty \leq \Delta, \quad \|A\|_\infty \leq \Delta$

$(m \cdot \Delta)^{O(m^2)}$ — Papadimitriou 1981

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Fact: There exists optimal solution x^* with

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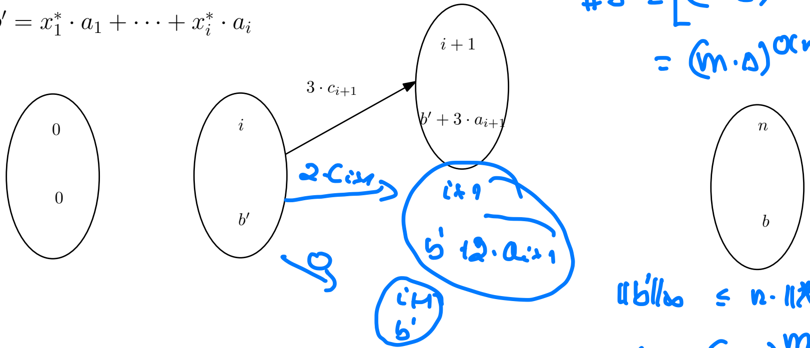
$$\# b' \leq \left[(m \cdot \Delta)^{O(m)}\right]^m = (m \cdot \Delta)^{O(m^2)}$$

$$b' = x_1^* \cdot a_1 + \cdots + x_i^* \cdot a_i$$



$$\|b'\|_\infty \leq n \cdot \|x^*\|_\infty \cdot \Delta \leq n \cdot (m \cdot \Delta)^m \cdot \Delta = (m \cdot \Delta)^{O(m)}$$

Number of Nodes approximately $(m \cdot \Delta)^{m^2}$

Number of nodes: $(m \cdot \Delta)^{O(m^2)}$

$(m \cdot \Delta)^{O(m^2)}$ — Papadimitriou 1981

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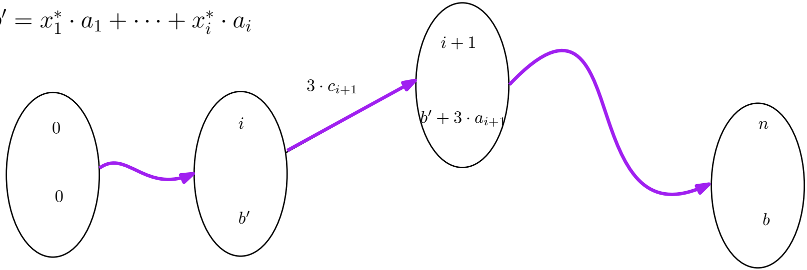


Find longest path.

Number of Nodes approximately $(m \cdot \Delta)^{m^2}$

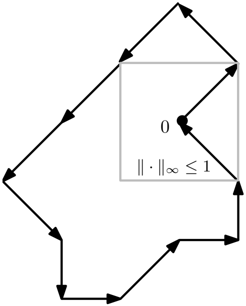
Running time $(m \cdot \Delta)^{O(m^2)}$

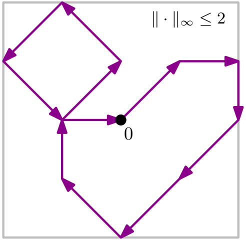
The Steinitz Lemma (Steinitz 1913)

$\max c^T \cdot x$
$Ax = b$
$x \geq 0, \; x \in \mathbb{Z}^n$

Let $x_1, \dots x_n \in \mathbb{R}^m$ such that $||x_i|| \leq 1$ for each i and $\sum_i x_i = 0$, then there exists a permutation π such that

$$||\sum_{i=1}^{k} x_{\pi_i}|| \leq m \text{ for each } k.$$





$$\max c^T x$$
$$Ax = b$$
$$x \geq 0, \quad x \in \mathbb{Z}^n$$

$A \in \mathbb{Z}^{m \times n}$ $\quad b \in \mathbb{Z}^m$

Def: Cycle of A: $x \in \mathbb{Z}^n_{\geq 0}$, $x \neq 0$ s.th.

$$Ax = 0$$

indecomposable if $x \neq x_1 + x_2$ for any cycles x_1, x_2.

Thm: Let $x \in \mathbb{Z}^n_{\geq 0}$ be an indecomposable cycle of A.

Then $\|x\|_1 \leq \boxed{(2m\Delta+1)^m}$ $\quad (m \cdot \Delta)^{O(m)}$, $\|A\|_\infty \leq \Delta$.

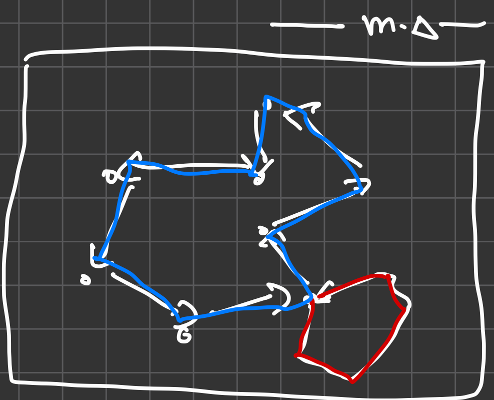
Proof: Cycle x, $\|x\|_1 = k$ corresponds to sequence.

$a_1 + a_2 + \dots + a_k = 0$ where a_i is col. of A.

j-th col. of A appears x_j times.

$\|a_i\|_\infty \leq \Delta$

$\exists$ reordering: s.th. $\left\| \sum_{i=1}^{q} a_{\pi_i} \right\|_\infty \leq m \cdot \Delta$



- Only visits integer points.
- does not re-visit an integer point.
- Upper bound ~~Estimate~~ $\|x\|_1$ of ~~x~~ by the number of integer points

$(2 \cdot m \cdot \Delta + 1)^m \Rightarrow \|x\|_1 \leq (2m\Delta+1)^m = (m \cdot \Delta)^{O(m)}$

IP $\quad \max \; c^T \cdot x$

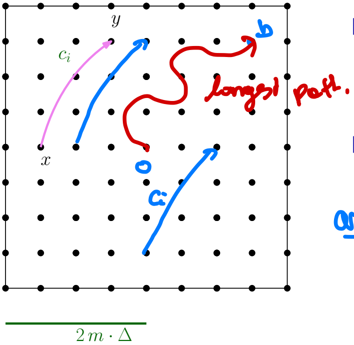
$$Ax = b$$
$$x \geq 0$$
$$x \in \mathbb{Z}^n$$

unbounded $\Leftrightarrow$ A has a cycle of positive cost w.r.t. $c^T \cdot x$

$(m \cdot \Delta)^{O(m)}$ — A smaller Steinitz state space $(m\cdot\Delta)^{O(m^2)}$

$$\begin{array}{lcc} \max & c^T x & \\ A\,x & = & b \\ x & \geq & 0 \\ x & \in & \mathbb{Z}^n \end{array}$$

Assumption:
$||A||_\infty, ||b||_\infty \leq \Delta$



- Let x^* be optimal solution
- **Steinitz sequence**: $a_{\pi_1} + \dots - b + \dots = 0$

$$0 = \underbrace{a_1 + \dots + a_1}_{x_1^* \text{ times}} + \quad \cdots \quad + \underbrace{a_n + \dots + a_n}_{x_n^* \text{ times}} \quad - \quad b$$

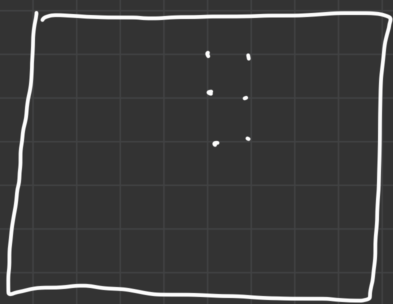
- Can be re-arranged such that partial sums of columns have $|| \cdot ||_\infty \leq 2\,m \cdot \Delta$
- $(m \cdot \Delta)^{O(m)}$ nodes

$m\cdot\Delta + \Delta$ $(m+1)\cdot\Delta$

arc: (x,y) iff $(y-x)$ is col. of A.

$y - x = a_i$

$2m\Delta$



\# integer points

$$\leq (4m\Delta + 1)^m = (m \cdot \Delta)^{O(m)}$$

$\boxed{\#\text{Nodes}}$

 # Arcs:

$$\#\text{Nodes} \cdot \underbrace{\#\text{cols}}_{(m\Delta)^{O(m)}}$$

$\Rightarrow$ Total size of graph is $(m \cdot \Delta)^{O(m)}$

Theorem: IP

$$\max c^T x \quad Ax = b, \quad x \geq 0, \quad x \in \mathbb{Z}^n$$

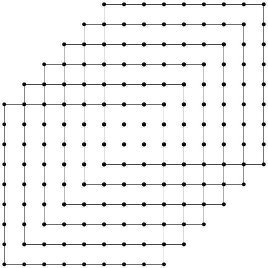
$\|A\|_\infty \leq \Delta$, $\|b\|_\infty \leq \Delta$, $A \in \mathbb{Z}^{m \times n}$

can be solved in time $(m \cdot \Delta)^{O(m)}$.

$(m \cdot \Delta)^{O(m)} \cdot ||b||_2$ — A milder dependence on b

$$\begin{array}{lcc} \max & c^T x & \\ A\,x & = & b \\ x & \geq & 0 \\ x & \in & \mathbb{Z}^n \end{array}$$

Assumption: Only $||A||_\infty \leq \Delta$



- Let x^* be optimal solution with $k = ||x^*||_1$
- **Steinitz sequence**:

$$\sigma = \underbrace{a_1 - b/k + \cdots + a_1 - b/k}_{x_1^* \text{ times}} + \quad \cdots \quad + \underbrace{a_n - b/k + \cdots + a_n - b/k}_{x_n^* \text{ times}}$$

- Can be re-arranged such that partial sums of columns have distance $|| \cdot ||_\infty \leq 2\,m \cdot \Delta$ from line-segment $0, b$
- $(m \cdot \Delta)^{O(m)} \cdot ||b||$ nodes

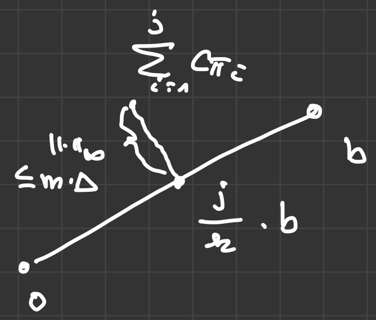
Theorem: *IP can be solved in time* $(m \cdot \Delta)^{O(m)} \cdot ||b||$.

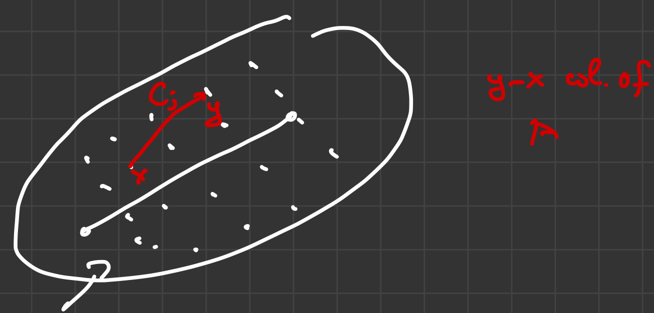
De-Arrangement:

$$\left(c_1 - \frac{b}{r}\right) + \left(c_2 - \frac{b}{r}\right) + \dots + \left(c_r - \frac{b}{r}\right) = 0$$

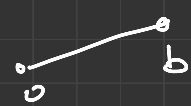
Steinitz. $\quad c_i$: ~~...~~ col of A, $i = 1, \dots, k$.

$$\underbrace{\Big\| \sum_{i=1}^{j} c_{\pi_i}}\; - \frac{j}{r} \cdot b \Big\|_\infty \overset{b}{\leq} m \cdot \Delta$$





integer points at distance $\leq m \cdot \Delta$ from line-segment



Compute longest path in graph.

Exercise: # of integer points at distance $\leq m \cdot \Delta$ from $0 \to b$ in ℓ_∞-norm is bounded by $(m \cdot \Delta)^{O(m)} \cdot \|b\|_2$

$$\max \ c^T \cdot x$$
$$Ax = b$$
$$0 \le x \le 1$$
$$x \in \mathbb{Z}^n$$

$A \in \{0,1\}^{m \times n}$

$n \le 2^m$ (no repeated cols.)

<u>Exercise:</u> can be solved in time $2^{O(m^2)}$

<u>Conjecture:</u> $2^{O(m)}$!

Proximity bounds

$$\begin{array}{rcl} \max & c^T x & \\ A\,x & = & b \\ & u \geq x \geq 0 & \\ & x \in \mathbb{Z}^n & \end{array}$$

x^* opt. sol of LP relaxation

There exists integer optimal solution z^* with

$$\|z^* - x^*\|_1 \leq n^2 \cdot (m \cdot \Delta)^m$$

(Cook et al. 1986)

$$\|z^* - x^*\|_1 \leq m \cdot (2\,m \cdot \Delta + 1)^m$$

generalization of a result of Aliev, Henk & Oertel (2017)

Lower bounds based on ETH

Exponential Time Hypothesis (ETH): There exists a constant $\epsilon > 0$ such that 3-SAT cannot be solved in time $2^{\epsilon \cdot n}$.

Theorem: For $0/1$*-matrices* A*, there is no* $2^{o(m \log m)}(n + ||b||_\infty)^{o(m)}$ *algorithm for IP, unless the ETH is false.*

(Knop, Pilipczuk, & Wrochna 2019)

IP with upper bounds on the variables

$$\begin{aligned} \max \quad & c^T x \\ A\,x \quad & = \quad b \\ 0 \leq\ & x \leq u \\ x \in\ & \mathbb{Z}^n \end{aligned}$$

$$(m \cdot \Delta)^{O(m^2)}$$

(E. & Weismantel 2018)

Open problem

Is there an $(m \cdot \Delta)^{O(m)}$ algorithm for IP with upper bounds on the variables? Are there matching lower bounds.

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$$(m \cdot \Delta)^{O(m)}$$

$$(m \cdot \Delta)^{O(m)} \cdot ||b||_\infty^2$$ (b unrestricted)

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$$(m \cdot \Delta)^{O(m)} \cdot ||b||_\infty^2$$ (b unrestricted)

$$\log(n)^{O(n)}$$ in the case $0 \leq x \leq n$

Open: $2^{O(n)}$